VERTEX COVERS AND SECURE DOMINATION IN GRAPHS

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Abstract. Let $G = (V,E)$ be a graph and let $S \subseteq V$. The set $S$ is a dominating set of $G$ if every vertex in $V \setminus S$ is adjacent to some vertex in $S$. The set $S$ is a secure dominating set of $G$ if for each $u \in V \setminus S$, there exists a vertex $v \in S$ such that $uv \in E$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G$. The minimum cardinality of a secure dominating set in $G$ is the secure domination number $\gamma_s(G)$ of $G$. We show that if $G$ is a connected graph of order $n$ with minimum degree at least two that is not a 5-cycle, then $\gamma_s(G) \leq n/2$ and this bound is sharp. Our proof uses a covering of a subset of $V(G)$ by vertex-disjoint copies of subgraphs each of which is isomorphic to $K_2$ or to an odd cycle.

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1. Introduction. In this paper, we continue the study of secure domination in graphs introduced by Cockayne, Grobler, Grundlingh, Munganga, and Van Vuuren [5] and explored further in [1, 2, 3, 4, 8] and elsewhere.

For notation and graph theory terminology we in general follow [6]. Specifically, let $G = (V,E)$ be a graph with vertex set $V$ of order $n$ and edge set $E$. We denote the degree of $v$ in $G$ by $d_G(v)$, or simply by $d(v)$ if the graph $G$ is clear from context. For a set $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. The (open) neighborhood of vertex $v \in V$ is denoted by $N(v) = \{u \in V \mid uv \in E\}$ while $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] =$

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The external private neighborhood epn(v, S) of a vertex v ∈ S is defined by epn(v, S) = \{u ∈ V \setminus S \mid N(u) \cap S = \{v\}\}. A cycle on n vertices is denoted by C_n, and a path on n vertices by P_n.

Two edges in a graph G are independent if they are not adjacent in G. A set of pairwise independent edges of G is called a matching in G, while a matching of maximum cardinality is a maximum matching. The number of edges in a maximum matching of G is called the matching number of G which we denote by \(\alpha'(G)\). A perfect matching in G is a matching with the property that every vertex is incident with an edge of the matching.

A set \(D \subseteq V\) is a dominating set if \(N[D] = V\). For sets \(A, B \subseteq V\), we say that \(A\) dominates \(B\) if \(B \subseteq N[A]\). The minimum cardinality of a dominating set is the domination number, denoted \(\gamma(G)\). Domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two excellent domination books by Haynes, Hedetniemi, and Slater [6, 7].

A set \(S \subseteq V\) is a secure dominating set, abbreviated SDS, if for each \(u \in V \setminus S\), there exists a vertex \(v\) such that

\[v \in N(u) \cap S \quad \text{and} \quad (S \setminus \{v\}) \cup \{u\}\] is a dominating set of \(G\). (1)

If \(u \in V \setminus S\) and \(v \in S\) satisfy (1), then we say that \(v\) securely dominates \(u\). We say that a vertex \(u\) is securely dominated by \(S\) if either \(u \in S\) or \(u \in V \setminus S\) and there is a vertex \(v \in S\) that securely dominates \(u\). Hence, \(S\) is a SDS of \(G\) if and only if every vertex of \(G\) is securely dominated by \(S\). We remark that every SDS of \(G\) is a dominating set of \(G\). The minimum cardinality of a SDS in \(G\) is the secure domination number of \(G\), denoted \(\gamma_s(G)\). A SDS of \(G\) of cardinality \(\gamma_s(G)\) is called a \(\gamma_s(G)\)-set. Secure domination in graphs has been studied in [1, 2, 3, 4, 5, 8] and elsewhere.

Cockayne, Favaron and Mynhardt [4] observed that for every graph, the secure domination number is at most the order of the graph minus the matching number. Since every SDS in a graph is a dominating set in the graph, the secure domination number is at least the domination number as first observed in [5]. Hence we have the following result.

**Proposition 1.** ([4, 5]) For every graph \(G\) of order \(n\), \(\gamma(G) \leq \gamma_s(G) \leq n - \alpha'(G)\).

## 2. Main result

Our aim in this paper is to determine an upper bound on the secure domination number of a connected graph in terms of its order. The star graphs \(K_{1,n-1}\) of order \(n \geq 2\) satisfy \(\gamma_s(G) = n - 1\). So henceforth we consider only connected graphs with minimum degree at least two. We shall prove:

**Theorem 1.** If \(G \neq C_5\) is a connected graph of order \(n\) with \(\delta(G) \geq 2\), then \(\gamma_s(G) \leq n/2\) and this bound is sharp.

### 2.1. Preliminary results and observations

The secure domination numbers of a path and a cycle are determined in [5].
Proposition 2. ([5]) (a) For $n \geq 1$, $\gamma_s(P_n) = \lceil \frac{3n}{7} \rceil$.
(b) For $n \geq 4$, $\gamma_s(C_n) = \gamma_s(P_n)$.

Observe that $\gamma_s(C_3) = 1$. Hence as a consequence of Proposition 2, we remark that if $n \geq 3$ is an odd integer and $n \neq 5$, then $\gamma_s(C_n) < \frac{n}{2}$.

Since adding edges to a graph does not increase its secure domination number, we observe that if $G_1, G_2, \ldots, G_r$ are vertex-disjoint subgraphs of a graph $G$ such that every vertex of $G$ belongs to (exactly) one of these subgraphs, then

$$\gamma_s(G) \leq \sum_{i=1}^{r} \gamma_s(G_i).$$

We shall need the following observation.

Observation 1. Let $G_1$ and $G_2$ be vertex-disjoint subgraphs of a graph $G$ such that $V(G) = V(G_1) \cup V(G_2)$. If $S_1 \subseteq V(G_1)$ is a SDS of $G_1$ and $S_2 \subseteq V(G_2)$ dominates $V(G_2)$ in $G_2$, then every vertex in $V(G_1)$ is securely dominated by $S = S_1 \cup S_2$ in $G$.

2.2. Proof of Theorem 1. To prove Theorem 1, we use a covering by vertex-disjoint copies of subgraphs each of which is isomorphic to $K_2$ or to an odd cycle. We shall need the following definition.

Definition 1. We define an odd-cycle-packing, abbreviated OCP, of a graph $G$ to be a covering of a subset of $V(G)$ by vertex-disjoint subgraphs of $G$ each of which is isomorphic to $K_2$ or to an odd cycle. If $F$ is a $K_2$-subgraph in an OCP $S$, then we refer to the two vertices of $F$ as partners in $S$.

Before proceeding with a proof of Theorem 1, we introduce the following terminology. If $S$ is an OCP of a graph $G$, we call a $K_2$-subgraph in $S$ a $K_2$-piece and we call an odd-cycle-subgraph in $S$ a cycle-piece. A cycle-piece of order $k$ we call a $k$-piece.

We are now in a position to present a proof of Theorem 1.

Proof of Theorem 1. Among all OCPs of $G = (V,E)$, let $S$ be chosen so that

1. $|U|$ is maximized, where $U$ is the subset of $V$ covered by $S$.
2. Subject to (1), the number of $K_2$-pieces is maximized.

We proceed further with a series of claims.

Claim A. If $v \in V \setminus U$, then the neighbors of $v$ in $G$ belong to different $K_2$-pieces.

Proof. By Rule (1) of our choice of $S$, $V \setminus U$ is an independent set. If $v$ is adjacent to a vertex of an odd-piece $F$, then we can cover $V(F) \cup \{v\}$ with vertex-disjoint copies of $K_2$. If $v$ is adjacent to both vertices of a $K_2$-piece $F$, then we can cover $V(F) \cup \{v\}$ with a copy of $C_3$. In both cases, we obtain an OCP that covers $|U| + 1$ vertices, a contradiction. \(\square\)
Claim B. There is no edge in $G$ joining vertices that belong to different odd-pieces.

Proof. Let $F_1$ and $F_2$ be distinct odd-pieces. If there is an edge of $G$ joining $F_1$ and $F_2$, then we can cover $V(F_1) \cup V(F_2)$ with vertex-disjoint copies of $K_2$, contradicting Rule (2) of our choice of $S$. \hfill \square

Claim C. Every 5-piece $F$ contains a vertex $v$ that is adjacent to a vertex in $V \setminus V(F)$. Moreover, for any such vertex $v$, the vertices in $N(v) \setminus V(F)$ belong to different $K_2$-pieces.

Proof. If $F$ is not an induced subgraph in $G$, then we can cover $V(F)$ with vertex-disjoint copies of $K_2$ and $C_3$, contradicting Rule (2) of our choice of $S$. Hence, $F$ is an induced 5-cycle in $G$. Since $G \neq C_5$, the 5-piece $F$ contains a vertex $v$ that is adjacent to a vertex in $V \setminus V(F)$. By Claim A and Claim B, the vertices in $N(v) \setminus V(F)$ belong to $K_2$-pieces. If $v$ is adjacent to both vertices in the same $K_2$-piece $H$, then we can cover $V(F) \cup V(H)$ with two vertex-disjoint copies of $K_2$ and a vertex-disjoint copy of $C_3$, contradicting Rule (2) of our choice of $S$. Hence, the neighbors of $v$ not in $F$ belong to different $K_2$-pieces. \hfill \square

Claim D. If a vertex of a $K_2$-piece $F$ is adjacent to a vertex in $V \setminus U$, then the neighbors of its partner in $S$ belong to different $K_2$-pieces.

Proof. Let $V(F) = \{u,v\}$. Suppose both vertices of $F$ are adjacent to vertices in $V \setminus U$. Let $u$ be adjacent to $u' \in V \setminus U$ and $v$ adjacent to $v' \in V \setminus U$. By Claim A, $u' \neq v'$. We can cover $V(F) \cup \{u',v'\}$ with two vertex-disjoint copies of $K_2$ to obtain an OCP that covers $|U| + 2$ vertices, a contradiction. Hence at most one vertex of $F$ is adjacent to a vertex in $V \setminus U$. Suppose that $v$ adjacent to a vertex $v'$ in $V \setminus U$, and so the partner $u$ of $v$ is not adjacent to a vertex in $V \setminus U$. If $u$ is adjacent to a vertex in an odd-piece $H$, then we can cover $V(F) \cup V(H) \cup \{v'\}$ with vertex-disjoint copies of $K_2$. If $u$ is adjacent to both vertices in the same $K_2$-piece $F'$, then we can cover $V(F) \cup V(F') \cup \{v'\}$ with vertex-disjoint copies of $K_2$ and $K_3$. In both cases, we produce a new OCP that covers $|U| + 1$ vertices, a contradiction. Hence, the neighbors of $u$ belong to different $K_2$-pieces. \hfill \square

Claim E. If a vertex in a $K_2$-piece $F$ is adjacent to a vertex from a 5-piece $H$, then its partner in $S$ is adjacent only to vertices that belong to $K_2$-pieces or to vertices of $H$.

Proof. Let $V(F) = \{u,v\}$. Suppose $v$ is adjacent to a vertex $v'$ from a 5-piece $H$. If $u$ is adjacent to a vertex $u' \in V \setminus U$, then we can cover $V(F) \cup V(H) \cup \{u'\}$ with vertex-disjoint copies of $K_2$. If $u$ is adjacent to a vertex $u'$ that belongs to an odd-piece $H'$ different from $H$, then we can cover $V(F) \cup V(H) \cup V(H')$ with
vertex-disjoint copies of $K_2$. In both cases, we produce a new OCP that covers $|U| + 1$ vertices, a contradiction. Hence, $u$ is adjacent only to vertices that belong to $K_2$-pieces or to vertices of $H$.

We now return to the proof of Theorem 1. We call a neighbor of a vertex in $V \setminus U$ an essential vertex. By Claim A and Claim D, the essential vertices belong to different $K_2$-pieces. Let $A_1$ be the set of all essential vertices; that is, $A_1 = N(V \setminus U)$.

For each 5-piece $F$, select one of its vertices $v$ that is adjacent to a vertex in a $K_2$-piece and call this vertex $v$ the link vertex of the 5-piece. By Claim C, the neighbors of $v$ not in $F$ belong to different $K_2$-pieces. A vertex in a $K_2$-piece that is adjacent to a link vertex of some 5-piece, we call an attacher. Let $A_2$ be the set of all attachers. By Claim E, the vertices in $A_2$ belong to different $K_2$-pieces. (Possibly, $A_1 \cap A_2 \neq \emptyset$.)

By Claim D, the neighbors of the partner of an essential vertex belong to different $K_2$-pieces. If the partner of an essential vertex is adjacent to no other vertex in $A_1 \cup A_2$, we call it a vulnerable vertex. Let $A_3$ be the set of all neighbors of vulnerable vertices different from the essential vertices that are their partners. Hence, if $u$ is a vulnerable vertex with partner $v$, then $N(u) \cap (A_1 \cup A_2) = \{v\}$ and $N(u) \setminus \{v\} \subseteq A_3$.

Let $A_4$ consist of one vertex from every $K_2$-piece that has no vertex in $A_1 \cup A_2 \cup A_3$. Hence if $v \in A_4$, then $v$ belongs to a $K_2$-piece, but neither $v$ nor its partner is an essential vertex or an attacher or is adjacent to a vulnerable vertex.

In each 5-piece $F$, we choose two vertices of $F$ as follows: Select a neighbor $u$ of the link vertex of $F$. The vertex $u$ and the vertex at distance 2 in $F$ from both $u$ and the link vertex we call the secure pair of $F$. Thus if $F$ is the cycle $v_1v_2v_3v_4v_5v_1$ with link vertex $v_1$, then the secure pair of $F$ is either the pair $\{v_2, v_4\}$ or the pair $\{v_3, v_5\}$ (but not both). Let $B$ be the set of all vertices of $G$ that belong to a secure pair.

If $F$ is an odd-piece that is not a 5-piece, let $C_F$ be a minimum SDS in $F$. Thus, $C_F$ is a $\gamma_2(F)$-set. Let $C$ be the union of the sets $C_F$ over all such odd-pieces $F$ that are not 5-pieces. From the remark mentioned in Section 2.1 concerning odd cycles, $\gamma_2(F) < |V(F)|/2$. Hence if $G_{odd}$ denotes the union of all odd-pieces that are not 5-cycles, then $C$ is a SDS of $G_{odd}$ and $|C| < |V(G_{odd})|/2$. Finally, let

$$A = \bigcup_{i=1}^{4} A_i \quad \text{and} \quad D = A \cup B \cup C.$$  

Claim F. The set $D$ is a dominating set of $G$. Further if $F$ is a subgraph in $S$, then the set $D \cap V(F)$ dominates $V(F)$.

Proof. By construction of the set $D$, if $F$ is a subgraph in $S$, then the set $D \cap V(F)$ dominates $V(F)$. Further by construction, every neighbor of a vertex in $V \setminus U$ belongs to $D$. Consequently, $D$ is a dominating set of $G$. \qed
Claim G. The set $A$ contains exactly one vertex from every $K_2$-piece.

Proof. Let $F$ be a $K_2$-piece and let $V(F) = \{u, v\}$. Suppose that $\{u, v\} \subseteq A$. Then, $\{u, v\} \subseteq A_1 \cup A_2 \cup A_3$.

Suppose $u$ or $v$, say $v$, is an essential vertex, i.e., $v \in A_1$. Let $v$ be adjacent to $v' \in V \setminus U$. By Claim D, the partner $u$ of $v$ is neither an essential vertex nor an attacher. Hence, $u \notin A_1 \cup A_2$, and so $u \in A_3$. Thus, $u$ is adjacent to a vulnerable vertex $x$. Let $y$ be the partner of $x$. Then, $y$ is an essential vertex and is adjacent to a vertex $y' \in V \setminus U$. If $v' = y'$, then we can cover $\{u, v, v', x, y\}$ with a 5-cycle. If $v' \neq y'$, then we can cover $\{u, v, v', x, y, y'\}$ with vertex-disjoint copies of $K_2$. In both cases, we produce a new OCP that covers more than $|U|$ vertices, a contradiction. Hence neither $u$ nor $v$ is an essential vertex, i.e., $\{u, v\} \subseteq A_2 \cup A_3$.

Suppose $u$ or $v$, say $v$, is an attacher, i.e., $v \in A_2$. Let $v'$ be a link vertex in a 5-piece $F$ that is adjacent to $v$. By Claim E, the partner $u$ of $v$ is not an attacher, and so $u \in A_3$. Thus, $u$ is adjacent to a vulnerable vertex $x$. Let $y$ be the partner of $x$. Then, $y$ is an essential vertex and is adjacent to a vertex $y' \in V \setminus U$. But then we can cover $V(F) \cup \{u, v, x, y, y'\}$ with vertex-disjoint copies of $K_2$ to produce a new OCP that covers $|U| + 1$ vertices, a contradiction. Hence neither $u$ nor $v$ is an attacher, i.e., $\{u, v\} \subseteq A_3$. Thus both $u$ and $v$ are adjacent to vulnerable vertices.

Let $u$ be adjacent to a vulnerable vertex $x$ with partner $y$, and let $v$ be adjacent to a vulnerable vertex $w$ with partner $z$. By Claim D, $x \neq w$. By definition of a vulnerable vertex, $y$ and $z$ are essential vertices. Thus, $y$ is adjacent to a vertex $y' \in V \setminus U$ and $z$ is adjacent to a vertex $z' \in V \setminus U$. If $y' = z'$, then we can cover $\{u, v, w, x, y, y', z\}$ with a 7-cycle. If $y' \neq z'$, then we can cover $\{u, v, w, x, y', z, z'\}$ with vertex-disjoint copies of $K_2$. In both cases, we produce a new OCP that covers more than $|U|$ vertices, a contradiction. We deduce, therefore, that our assumption that $\{u, v\} \subseteq A$ is incorrect. Hence, $A$ contains at most one of $u$ and $v$. However by construction, the set $A$ contains at least one vertex from every $K_2$-piece. Thus, $A$ contains exactly one of $u$ and $v$. 

Claim H. Every vertex in $V \setminus U$ is securely dominated by the set $D$.

Proof. Let $v' \in V \setminus U$ and let $v \in N(v')$. Then, $v$ is an essential vertex, and so $v \in A_1$. By construction, the partner $u$ of $v$ is either adjacent to at least one vertex in $(A_1 \cup A_2) \setminus \{v\}$ or is a vulnerable vertex, in which case $N(u) \setminus \{v\} \subseteq A_3$. In both cases, the partner $u$ of $v$ is adjacent to a vertex in $A \setminus \{v\}$. Hence by Claim F, the set $D \setminus \{v\}$ dominates $V \setminus \{v, v'\}$, and so the set $(D \setminus \{v\}) \cup \{u'\}$ is a SDS of $G$, i.e., $v$ $D$-defends $v'$. Thus, $v'$ is securely dominated by the set $D$.

Claim I. Every vertex in a $K_2$-piece is securely dominated by the set $D$.

Proof. Let $F$ be a $K_2$-piece and let $V(F) = \{u, v\}$. By Claim G, the set $A$ contains exactly one of $u$ and $v$. We may assume $v \in A \subseteq D$. Thus, $v$ is securely dominated by $D$. We show that $u$ is also securely dominated by $D$. Every neighbor of a vertex
in \( V \setminus U \) belongs to \( A_1 \), and so every vertex in \( V \setminus U \) is dominated by at least two vertices in \( D \). By Claim F, if \( F \) is a subgraph in \( S \), then the set \( D \cap V(F) \) dominates \( V(F) \). Hence the set \( D \setminus \{ v \} \) dominates \( V \setminus \{ u, v \} \), and so the set \( (D \setminus \{ v \}) \cup \{ u \} \) is a SDS of \( G \), i.e., \( v \) \( D \)-defends \( u \). Thus, \( u \) is securely dominated by the set \( D \).

**Claim J.** Every vertex in an odd-piece that is not a 5-piece is securely dominated by the set \( D \).

**Proof.** Let \( G_{\text{odd}} \) denote the union of all odd-pieces that are not 5-cycles. By definition, the set \( C \subseteq D \) is a SDS of \( G_{\text{odd}} \). Let \( V_{\text{odd}} = V(G_{\text{odd}}) \). Let \( G_1 = G[V_{\text{odd}}] \) and let \( G_2 = G - V_{\text{odd}} \). Since adding edges to a graph does not increase its secure domination number, the set \( C \) is a SDS of \( G_1 \). By Claim F, the set \( A \cup B \) dominates \( V(G_2) \). Hence, by Observation 1, every vertex in \( V(G_1) = V_{\text{odd}} \) is securely dominated by the set \( D \) in \( G \).

**Claim K.** Every vertex in a 5-piece is securely dominated by the set \( D \).

**Proof.** Let \( F : v_1v_2v_3v_4v_5v_1 \) be a 5-piece with link vertex \( v_1 \). Without loss of generality, we may assume that \( \{v_2, v_4\} \) is the secure pair of \( F \), and so \( D \setminus V(F) = B \cup V(F) = \{v_2, v_4\} \). We remark that the link vertex \( v_1 \) is dominated by at least one vertex in \( A \). Let \( v \in V(F) \). If \( v = v_1 \), then \( v_2 \) \( D \)-defends \( v \). If \( v = v_3 \), then \( v_2 \) \( D \)-defends \( v \). If \( v = v_5 \), then \( v_4 \) \( D \)-defends \( v \). Hence every vertex of \( F \) is securely dominated by the set \( D \).

As an immediate consequence of Claims H, I, J and K, we have the following result.

**Claim L.** The set \( D \) is a SDS of \( G \).

**Proof.** Let \( V_A \) be the set of all vertices that belong to some \( K_2 \)-piece. Let \( V_B \) be the set of all vertices that belong to some 5-piece. Let \( V_C \) be the set of all vertices that belong to some odd-piece that is not a 5-piece. Then, \((V_A, V_B, V_C)\) is a weak partition of the set \( U \) (where by a weak partition of a set we mean a partition of the set in which some of the subsets may be empty). By Claim G and by the way in which the sets \( B \) and \( C \) are defined, we have that \( |A| = |V_A|/2 \), \( |B| = 2|V_B|/5 \) and \( |C| < |V_C|/2 \). Consequently, by Claim L,

\[
\gamma_s(G) \leq |D| = |A| + |B| + |C| \leq \frac{1}{2}|V_A| + \frac{2}{5}|V_B| + \frac{1}{2}|V_C| \leq \frac{1}{2}|V| = \frac{n}{2}
\]

with equality if and only if \( V = A \). Hence, \( \gamma_s(G) \leq |D| \leq n/2 \). Furthermore, if \( \gamma_s(G) = n/2 \), then \( V = A \), i.e., \( G \) has a perfect matching. This completes the proof of Theorem 1. \( \square \)
2.3. Sharpness of Theorem 1. In this section we show that the bound established in Theorem 1 is sharp. For this purpose, we construct a family $G$ of connected graphs with minimum degree two and secure domination number exactly one-half their order. Let $G$ denote the family of all connected graphs $G$ that are obtained from the disjoint union of 4-cycles and 6-cycles by taking a red-blue coloring of the vertices and then adding some or all of the edges joining green vertices. A graph in the family $G$ constructed from five copies of $C_4$ and one copy of $C_6$ is shown in Figure 1 with the green vertices indicated by the large vertices.

![Figure 1: A graph $G$ in the family $G$.](image)

**Proposition 3.** If $G \in G$ has order $n$, then $G$ is a connected graph with $\delta(G) \geq 2$ satisfying $\gamma_s(G) = n/2$.

**Proof.** By construction, $G$ is a connected graph with $\delta(G) \geq 2$. Let $G$ be constructed from $r$ copies of $C_4$ and $s$ copies of $C_6$. Then, $n = 4r + 6s$. Let $S$ be a SDS of $G$. We show that $S$ contains at least two vertices from every copy of $C_4$ and three vertices from every copy of $C_6$ that was used to construct $G$.

Let $F_1: a_1, a_2, a_3, a_4, a_1$ be a $C_4$ that was used to construct $G$, where $a_1$ and $a_3$ are colored green. Then, $d_G(a_2) = d_G(a_4) = 2$, while $d_G(a_1) \geq 2$ and $d_G(a_3) \geq 2$ (possibly, $a_1$ and $a_3$ are adjacent). Since $S$ is a dominating set of $G$, $S$ contains at least one vertex of $F_1$. Suppose $|S \cap V(F_1)| = 1$. Then in order to dominate $a_2$ and $a_4$, either $a_1 \in S$ or $a_3 \in S$. But then neither $a_2$ nor $a_4$ is $S$-defended by a vertex of $S$, contradicting the fact that $S$ is a SDS of $G$. Hence, $|S \cap V(F_1)| \geq 2$.

Let $F_2: b_1, b_2, b_3, b_4, b_5, b_6, b_1$ be a $C_6$ that was used to construct $G$, where $b_1$, $b_3$ and $b_5$ are colored green. Then each of $b_2$, $b_4$ and $b_6$ has degree 2 in $G$, while each of $b_1$, $b_3$ and $b_5$ has degree at least 2 in $G$ (some or none of the edges $b_1b_3$, $b_1b_5$ and $b_3b_5$ may be present in $G$). Since $S$ is a dominating set of $G$, $S$ contains at least two vertices of $F_2$. Suppose $|S \cap V(F_2)| = 2$. Since $S$ dominates $\{b_2, b_4, b_6\}$, $S$ must contain at least one attacher of $F_2$. We may assume that $b_1 \in S$. In order to dominate $b_4$, the remaining vertex in $S \cap V(F_2)$ belongs to the set $\{b_1, b_3, b_5\}$. If $b_4 \in S$, then there is no vertex of $S$ that $S$-defends $b_2$ or $b_6$. If $b_3 \in S$, then there is no vertex of $S$ that $S$-defends $b_2$. If $b_3 \in S$, then there is no vertex of $S$ that $S$-defends $b_6$. Since all three cases produce a contradiction, we deduce that $|S \cap V(F_2)| \geq 3$.

Hence we have shown that $S$ contains at least two vertices from every copy of $C_4$ and at least three vertices from every copy of $C_6$ that was used to construct
Thus, \(|S| \geq 2r + 3s = n/2\), and so \(\gamma_s(G) \geq n/2\). However the set of \(2r + 3s\) green vertices in \(G\) form a secure dominating set of \(G\), and so \(\gamma_s(G) \leq n/2\). Consequently, \(\gamma_s(G) = n/2\).

3. Closing remarks. It remains an open problem to characterize the graphs that achieve equality in the bound of Theorem 1.

**Problem 1.** Characterize the connected graphs \(G\) with minimum degree \(\delta(G) \geq 2\) and order \(n\) satisfying \(\gamma_s(G) = n/2\).

As shown in Proposition 3, there is an infinite family of graphs that achieve equality in the bound of Theorem 1. Observe that for \(r \in \{2, 3, 4\}\), \(\gamma_s(K_{r,r}) = r\). Hence there are graphs with minimum degree \(\delta = 3\) or \(\delta = 4\) that achieve the upper bound of Theorem 1. For a graph \(G\) with minimum degree \(\delta \geq 5\) and order \(n\), it remains an open problem to find a sharp upper bound on \(\gamma_s(G)\) in terms of \(\delta\) and \(n\).

**References**


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