The mixed irredundant Ramsey numbers
\[ t(3, 7) = 18 \text{ and } t(3, 8) = 22 \]

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Abstract

The mixed irredundant Ramsey number \( t(m, n) \) is the smallest natural number \( t \) such that if the edges of the complete graph \( K_t \) on \( t \) vertices are arbitrarily bi-coloured using the colours blue and red, then necessarily either the subgraph induced by the blue edges has an irredundant set of cardinality \( m \) or the subgraph induced by the red edges has an independent set of cardinality \( n \) (or both). Previously it was known that \( 18 \leq t(3, 7) \leq 22 \) and \( 18 \leq t(3, 8) \leq 28 \). In this paper we prove that \( t(3, 7) = 18 \) and \( t(3, 8) = 22 \).

Keywords: Irredundance, Ramsey number.

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1 Introduction

Let \( G = (V_G, E_G) \) be a simple graph. A set of vertices \( I \subset V_G \) is said to be independent if no two vertices in \( I \) are adjacent in \( G \). The closed neighbourhood of a vertex \( v \in V_G \), denoted by \( N[v] \), is defined as the subset of all vertices in \( V_G \) that are adjacent to \( v \) in \( G \), together with the vertex \( v \) itself. The closed neighbourhood of a set of vertices \( S = \{v_1, \ldots, v_m\} \subset V_G \), denoted by \( N[S] \), is simply defined as \( N[S] = \cup_{i=1}^{m} N[v_i] \). The set of private neighbours of a vertex \( v \in V_G \) with respect to some subset \( S \subset V_G \) is defined as \( \text{PN}(v, S) = N[v] \setminus N[S \setminus \{v\}] \). Finally, a set of vertices \( X \subset V_G \) is said to be irredundant if every vertex \( v \in X \) has at least one private neighbour in \( G \) with respect to \( X \) (that is, if \( \text{PN}(v, X) \neq \emptyset \) for all \( v \in X \)).

We call a bi-colouring of the edges of a graph, using the colours red and blue, a red-blue edge colouring of the graph. If \( R \) and \( B \) are the subgraphs induced by respectively the

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red and the blue edges of such a colouring, the colouring is denoted by the ordered pair \((R, B)\), and \(R\) is referred to as the red subgraph, while \(B\) is called the blue subgraph.

The following Ramsey numbers have previously been studied in the literature (amongst others):

The independent Ramsey number \(r = r(m, n)\) is the smallest natural number \(r\) such that in any red-blue edge colouring \((R, B)\) of the complete graph \(K_r\) of order \(r\), there is an independent set of cardinality \(m\) in the blue subgraph \(B\) or an independent set of cardinality \(n\) in the red subgraph \(R\). This definition may be traced back to the celebrated theorem by Ramsey [16].

The irredundant Ramsey number \(s = s(m, n)\) is the smallest natural number \(s\) such that in any red-blue edge colouring \((R, B)\) of the complete graph \(K_s\) of order \(s\), there is an irredundant set of cardinality \(m\) in the blue subgraph \(B\) or an irredundant set of cardinality \(n\) in the red subgraph \(R\). This definition is due to Brewster et al. [1].

The mixed irredundant Ramsey number \(t = t(m, n)\) is the smallest natural number \(t\) such that in any red-blue edge colouring \((R, B)\) of the complete graph \(K_t\) of order \(t\), there is an irredundant set of cardinality \(m\) in the blue subgraph \(B\) or an independent set of cardinality \(n\) in the red subgraph \(R\). This definition is due to Cockayne et al. [5].

It is a well-known fact that every independent set of any simple graph \(G\) is also an irredundant set of \(G\) [11]. Therefore the inequality chain

\[
s(m, n) \leq t(m, n) \leq r(m, n) \tag{1.1}
\]

holds for all \(m, n \geq 2\). Besides the trivial Ramsey numbers \(r(2, n) = s(2, n) = t(2, n) = n\) for all \(n \geq 2\), the known values of the Ramsey numbers defined above are shown in Table 1.1.

<table>
<thead>
<tr>
<th>Irredundant number</th>
<th>Source</th>
<th>Mixed irredundant number</th>
<th>Source</th>
<th>Classical number</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s(3, 3) = 6)</td>
<td>[1]</td>
<td>(t(3, 3) = 6)</td>
<td>[5]</td>
<td>(r(3, 3) = 6)</td>
<td>[8]</td>
</tr>
<tr>
<td>(s(3, 4) = 8)</td>
<td>[1]</td>
<td>(t(4, 3) = 8)</td>
<td>[5]</td>
<td>(r(3, 4) = 9)</td>
<td>[8]</td>
</tr>
<tr>
<td>(s(3, 5) = 12)</td>
<td>[1]</td>
<td>(t(3, 4) = 9)</td>
<td>[5]</td>
<td>(r(3, 5) = 14)</td>
<td>[8]</td>
</tr>
<tr>
<td>(s(3, 6) = 15)</td>
<td>[2]</td>
<td>(t(3, 5) = 12)</td>
<td>[5]</td>
<td>(r(3, 6) = 18)</td>
<td>[7]</td>
</tr>
<tr>
<td>(s(3, 7) = 18)</td>
<td>[4]</td>
<td>(t(5, 3) = 13)</td>
<td>[5]</td>
<td>(r(3, 7) = 23)</td>
<td>[13]</td>
</tr>
<tr>
<td>(s(4, 4) = 13)</td>
<td>[6]</td>
<td>(t(3, 6) = 15)</td>
<td>[12]</td>
<td>(r(3, 8) = 28)</td>
<td>[14]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(t(6, 3) = 17)</td>
<td>[3]</td>
<td>(r(3, 9) = 36)</td>
<td>[9]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(t(4, 4) = 14)</td>
<td>[3]</td>
<td>(r(4, 4) = 18)</td>
<td>[8]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(r(4, 5) = 25)</td>
<td>[15]</td>
</tr>
</tbody>
</table>

Table 1.1: Ramsey numbers known exactly.

The growth property \(x(m, n) \leq x(m + 1, n)\) holds for any of the above Ramsey numbers \(x \in \{r, s, t\}\) and for all \(m, n \geq 2\), while the symmetry property \(x(m, n) = x(n, m)\) holds for the Ramsey numbers \(x \in \{r, s\}\) and for all \(m, n \geq 2\). In contrast, \(t(m, n) \neq t(n, m)\) in general. Finally, the following recursive upper bound is also well known.

**Proposition 1** \(x(m, n) \leq x(m - 1, n) + x(m, n - 1)\) for any \(x \in \{r, s, t\}\) and for all \(m, n \geq 2\). If both \(x(m - 1, n)\) and \(x(m, n - 1)\) are even, then strict inequality holds. ■
Therefore the mixed irredundant Ramsey numbers \( t(3, 7) \) and \( t(3, 8) \) are known to lie within the bounds

\[
18 = s(3, 7) \leq t(3, 7) \leq \min\{t(2, 7) + t(3, 6), r(3, 7)\} = \min\{7 + 15, 23\} = 22,
18 = s(3, 7) \leq s(3, 8) \leq t(3, 8) \leq \min\{t(2, 8) + t(3, 7), r(3, 8)\} = \min\{8 + 22, 28\} = 28.
\]

In this paper we show that, in fact, \( t(3, 7) = 18 \) and \( t(3, 8) = 22 \).

## 2 Preliminary results

An \((m, n, p)\)-colouring is a red-blue colouring \((R, B)\) of the edges of the complete graph \(K_p\) for which the subgraph \(B\) has no irredundant set of cardinality \(m\) and for which the subgraph \(R\) has no independent set of cardinality \(n\). The minimum degree of the subgraph \(R\) [\(B\), resp.] is referred to as the minimum red [blue, resp.] degree, and similarly for the maximum degree. The subgraph of the blue [red, resp.] subgraph of an \((m, n, p)\)-colouring induced by a set \(A \subseteq V_{K_p}\) is denoted by \(\langle A \rangle_{\text{blue}}\) [\(\langle A \rangle_{\text{red}}\), resp.]. Furthermore, \(D_1(v) [D_{>1}(v)\) or \(D_{\geq 1}(v), \text{resp.}\)] denotes the set of vertices at distance \(i \in \mathbb{N}\) [at a distance greater than \(i\) or at a distance at least \(i\), resp.] from some specified vertex \(v\) in the red subgraph of a \((3, n, p)\)-colouring. The following characterisation dates from 1989 and is due to Brewster et al. [1].

**Proposition 2 ([1])** The blue subgraph of a \((3, n, p)\)-colouring has an irredundant set of cardinality 3 if and only if there is a 3-cycle in the red subgraph or a 6-cycle \(v_1v_2v_3v_4v_5v_6v_1\) in the red subgraph with the edges \(v_1v_4\), \(v_2v_5\) and \(v_3v_6\) in the blue subgraph.

We call the second substructure in the above proposition a red 6-cycle with blue diagonals. Furthermore, the following useful results are due to Hattingh [10] and date from 1990.

**Proposition 3 ([10])** For any \((m, n, p)\)-colouring with minimum red degree \(\delta\) and maximum red degree \(\Delta\), and any \(m, n \geq 2\), \(p - s(m, n - 1) \leq \delta \leq \Delta \leq s(m - 1, n) - 1\).

**Proposition 4 ([10])** Let \(v\) be any vertex of a \((3, n, p)\)-colouring.
(a) If \(xyz\) is a path in \(\langle D_{>1}(v)\rangle_{\text{red}}\) and \(x, z \in D_2(v)\), then \(x\) and \(z\) are joined by means of red edges to a common vertex in \(D_1(v)\).
(b) If \(A \subseteq D_{>1}(v)\) contains at most one vertex of \(D_{>2}(v)\), then \(\langle A \rangle_{\text{red}}\) is bipartite.

We have the following corollary of Proposition 4(a).

**Corollary 1** If there is a star in \(\langle D_{>1}(v)\rangle_{\text{red}}\) with three end-vertices \(x_1, x_2, x_3 \in D_2(v)\), then \(x_1, x_2\) and \(x_3\) are joined by means of red edges to a common vertex in \(D_1(v)\).

**Proof**: By contradiction. If \(x_1, x_2\) and \(x_3\) are not joined by means of red edges to a common vertex in \(D_1(v)\), then each pair of vertices from the set \(\{x_1, x_2, x_3\}\) must have a distinct common neighbour in \(D_1(v)\) by Proposition 4(a). But then these three common
neighbours in $D_1(v)$ and $\{x_1, x_2, x_3\}$ form a red 6-cycle with blue diagonals in $(D_1(v) \cup D_2(v))$, a contradiction by Proposition 2. \[\blacksquare\]

Since $(D_2(v) \cup \{u\})_{\text{red}}$ is a bipartite graph for any $u \in D_{2\ell}(v)$ by Proposition 4(b), it follows that $(D_2(v))_{\text{red}}$ must itself be bipartite. In the remainder of the paper we use the symbol $c$ to denote the number of components of $(D_2(v))_{\text{red}}$ and we denote the bipartitions of these components by $(X_\ell, Y_\ell)$, for all $\ell = 1, \ldots, c$. We may assume, without loss of generality, that $|X_\ell| \geq |Y_\ell|$ for all $\ell = 1, \ldots, c$. Define $X = \cup_{\ell=1}^c X_\ell$ and $Y = \cup_{\ell=1}^c Y_\ell$. Then $|X| \geq |Y|$. We have the following six useful results.

**Lemma 1** Let $v$ be any vertex of a $(3, n, p)$-colouring and suppose $x \in D_{2\ell}(v)$.

(a) If $x$ sends a red edge to $X_\ell$, then $x$ sends no red edge to $Y_\ell$ and vice versa for any $\ell = 1, \ldots, c$.

(b) If $|X| \geq n - 2$ and there is exactly one $\ell \in \{1, \ldots, c\}$ such that $|X_\ell| > |Y_\ell|$, then each vertex in $D_3(v)$ sends a red edge to $X_\ell$.

(c) If $|Y| \geq n - 3$, then there exists, for each edge $uv$ in $(D_3(v))_{\text{blue}}$, an $\ell \in \{1, \ldots, c\}$ such that $u$ sends a red edge to $X_\ell$ and $w$ sends a red edge to $Y_\ell$.

(d) If $|Y| \geq n - 3$ and there is an odd cycle in $(D_{2\ell})_{\text{blue}}$, then the pairs of red edges sent to $D_2(v)$ by the edges of this cycle according to part (c) above go to at least two components of the bipartite graph $(D_2)_{\text{red}}$.

(e) If $|Y| \geq n - 3$, $\Delta(R) = 4$ and $Z$ is a partite set of a component of $(D_2(v))_{\text{red}}$ such that $|Z| \geq 2$, then any vertex $z \in Z$ sends at most one red edge to $D_3(v)$.

(f) If $|X| \geq n - 2$, then $D_{2\ell}(v) = \emptyset$.

**Proof:** (a) If the statement of the lemma is false, then an odd cycle results in the bipartite graph $(D_2(v) \cup \{x\})_{\text{red}}$ guaranteed by Proposition 4(b), a contradiction.

(b) Suppose $|X| \geq n - 2$ and that there is exactly one $\ell \in \{1, \ldots, c\}$ such that $|X_\ell| > |Y_\ell|$ and that there is a vertex $u \in D_3(v)$ sending no red edge to $X_\ell$. It follows by part (a) that $u$ does not send a red edge to both $X_i$ and $Y_i$, for all $i = 1, \ldots, c$. Therefore we may select from $X_i$ or $Y_i$ the part, $Z_i$ (say), from each component $i \in \{1, \ldots, c\}$ sending no red edge to $u$. But then $|\cup_{i=1}^c Z_i| = \sum_{i=1}^c |X_i| \geq n - 2$ and hence $\{u, v\} \cup (\cup_{i=1}^c Z_i)$ is an independent set of cardinality at least $n$ in the red subgraph of the $(3, n, p)$-colouring, a contradiction.

(c) Suppose $|Y| \geq n - 3$ and that there is a blue edge $uv$ in $D_3(v)$, but no $i \in \{1, \ldots, c\}$ for which $u$ sends a red edge to $X_i$ and $w$ sends a red edge to $Y_i$. Then we may select from $X_i$ or $Y_i$ the part, $Z_i$ (say), from each component $i \in \{1, \ldots, c\}$ sending no red edge to either $u$ or to $w$ by part (a). But then $|\cup_{i=1}^c Z_i| \geq \sum_{i=1}^c |Y_i| \geq n - 3$ and hence $\{u, v, w\} \cup (\cup_{i=1}^c Z_i)$ is an independent set of cardinality at least $n$ in the red subgraph of the $(3, n, p)$-colouring, a contradiction.

(d) Suppose all the pairs of red edges sent to $D_2(v)$ by the edges of the odd blue cycle in $(D_{2\ell})$ according to part (c) above go to the same component, $(X', Y')$ (say), of $D_2(v)$. Then it follows by part (a) above that each vertex of the blue cycle in $(D_{2\ell})$ sends red edges to either $X'$ or $Y'$, but not to both. Therefore, the vertices of the blue cycle in $(D_{2\ell})$ send red edges to $X'$ or $Y'$ in alternating fashion as one traverses the blue cycle, but this is impossible, since the blue cycle is odd.
(e) Suppose $Z$ and $Z'$ are the partite sets of a component of the bipartite graph $\langle D_2(v)\rangle_{\text{red}}$ such that $|Z| \geq 2$. Since $\langle Z\rangle_{\text{red}}$ is connected, there is a red path $z_1z'_2 \langle D_2(v)\rangle_{\text{red}}$ with $z_1, z_2 \in Z$ and $z'_2 \in Z'$. If $z_1$ is joined by means of red edges to two vertices $w, w' \in D_3(v)$, then $ww'$ is a blue edge (in order to avoid the formation of a red triangle). But then we may assume by part (c) that the blue edge $ww'$ sends a red edge $wx$ to a vertex $x \in X_\ell$, and another red edge $w'y$ to a vertex $y \in Y_\ell$. Moreover, $x, y \notin Z$ by part (a). It also follows by Proposition 4(a) that each pair of endpoints of the red paths $xwz_1, z_1w'y$ and $z_2z'_1$ must each have a (not necessarily distinct) common neighbour in $D_1(v)$, but this is a contradiction, because then $z_1$ or one of these common neighbours will have a red degree larger than $\Delta(R) = 4$.

(f) Suppose $|X| \geq n - 2$ and that $u \in D_{>3}(v)$. Then $\{u, v\} \cup X$ is an independent set of cardinality at least $n$ in the red subgraph of the $(3, n, p)$-colouring, a contradiction. ■

By combining the results of Lemma 1 we have the following useful result.

**Corollary 2** If $|X| \geq n - 2$, $|Y| \geq n - 3$ and there is exactly one $\ell \in \{1, \ldots, c\}$ such that $|X_\ell| > |Y_\ell|$, then

(a) the pair of red edges sent by any edge in $\langle D_3(v)\rangle_{\text{blue}}$ to $D_3(v)$ necessarily goes to a balanced component of $\langle D_2(v)\rangle_{\text{red}}$, i.e. not to the component $(X_\ell, Y_\ell)$.

(b) $|X_\ell| \geq |D_3(v)|$ if $\Delta(R) = 4$ and $|X_\ell| \geq 2$.

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Suppose there exists a $(3, 7, 18)$-colouring. Let $\Lambda$ and $\overline{\Lambda}$ be respectively the red and blue subgraphs of this colouring, and denote the minimum and maximum red degrees of $\Lambda$ by $\delta(\Lambda)$ and $\Delta(\Lambda)$, respectively. Suppose $\lambda$ is a vertex of red degree $\Delta(\Lambda)$ in this colouring. Since $t(2, 7) = 7$ and $t(3, 6) = 15$, it follows by Proposition 3 that

$$3 \leq \delta(\Lambda) \leq \Delta(\Lambda) \leq 6. \quad (3.1)$$

It is, however, possible to improve the bounds on $\Delta(\Lambda)$ in (3.1).

**Lemma 2** $3 \leq \delta(\Lambda) \leq \Delta(\Lambda) \leq 4$.

**Proof:** Suppose first that $\Delta(\Lambda) = 6$. Then $D_{>2}(\lambda) = \emptyset$, for the existence of an element $v \in D_{>2}(\lambda)$ would induce an independent set $\{v\} \cup D_{>2}(\lambda)$ of cardinality 7 in $\Lambda$. It follows by Proposition 4(b) that $\langle D_2(\lambda)\rangle_{\text{red}}$ is bipartite and since $|D_2(\lambda)| = 11$, this bipartite graph has a partite set, $Z$ (say), of cardinality at least 6. But then $\{\lambda\} \cup Z$ is an independent set of cardinality at least 7 in $\Lambda$, a contradiction showing that $\Delta(\Lambda) < 6$.

Suppose next that $\Delta(\Lambda) = 5$. If $\langle D_{>2}(\lambda)\rangle_{\text{red}}$ has two independent vertices $u$ and $v$ (say), then $\{u, v\} \cup D_1(\lambda)$ is an independent set of cardinality 7 in $\Lambda$, a contradiction. Hence, if $|D_{>2}(\lambda)| \geq 3$, then $D_{>2}(\lambda)$ induces a red triangle in $\Lambda$. Therefore $|D_{>2}(\lambda)| \leq 2$, and so $|D_2(\lambda)| = 18 - 1 - 5 - |D_{>2}(\lambda)| \geq 10$. Since $\langle D_2(\lambda) \cup \{w\}\rangle_{\text{red}}$ is bipartite for any vertex $w \in D_{>2}(\lambda)$ by Proposition 4(b), it must have a partite set, $Z'$ (say), of cardinality at
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Figure 3.1: If \( \Delta(\Lambda) = 3 \), then \( \langle D_{>2}(\lambda) \rangle_{\text{red}} \) is isomorphic to \( E_{10} \) or \( E_{11} \) [3, Table 5].

least 6. But then \( \{\lambda\} \cup Z' \) is an independent set of cardinality at least 7 in \( \Lambda \), again a contradiction, showing that \( \Delta(\Lambda) < 5 \). □

We show next that \( \Lambda \) is not cubic.

**Lemma 3** \( \Delta(\Lambda) = 4 \).

**Proof:** Suppose \( \Delta(\Lambda) = 3 \). Then it follows by Lemma 2 that \( \Lambda \) is 3-regular, which means that \( |D_2(\lambda)| \leq 6 \). However, if \( |D_2(\lambda)| < 6 \), then \( |D_{>2}(\lambda)| \geq 9 \), and since \( t(3, 4) = 9 \), it follows that there is an irredundant set \( Z \) of cardinality 3 in \( \langle D_{>2}(\lambda) \rangle_{\text{blue}} \) or an independent set \( Z' \) of cardinality 4 in \( \langle D_{>2}(\lambda) \rangle_{\text{red}} \). In the former case \( Z \) is also an irredundant set of cardinality 3 in \( \overline{\Lambda} \) (a contradiction), while in the latter case \( Z' \cup D_1(\lambda) \) is an independent set of cardinality 7 in \( \Lambda \) (again a contradiction). We conclude that \( |D_2(\lambda)| = 6 \) and hence that \( |D_{>2}(\lambda)| = 8 \).

According to [3, Table 5] \( \langle D_{>2}(\lambda) \rangle_{\text{red}} \) must therefore be isomorphic to the red subgraph of one of only two possible \((3, 4, 8)\)-colourings — these red subgraphs \( E_{10} \) and \( E_{11} \) are shown in Figure 3.1. Clearly, \( \langle D_{>2}(\lambda) \rangle_{\text{red}} \not\cong E_{11} \) because at least one vertex of \( D_{>2}(\lambda) \) must be adjacent to a vertex of \( D_2(\lambda) \), but \( E_{11} \) is already cubic. If \( \langle D_{>2}(\lambda) \rangle_{\text{red}} \cong E_{10} \), then, since each vertex \( D_2(\lambda) \) is adjacent to a vertex in \( D_1(\lambda) \) and \( \Lambda \) is cubic, there is only one way to draw the edges between \( D_1(\lambda) \) and \( D_2(\lambda) \), as shown in Figure 3.2. Also, since \( \Lambda \) is cubic, all vertices of degree 2 in \( E_{10} \) must be in \( D_3(\lambda) \), and all vertices of degree 3 in \( E_{10} \) must be in \( D_4(\lambda) \), as shown in Figure 3.2. Since \( y_1 \) is adjacent to exactly one vertex in \( D_2(\lambda) \), we may assume without loss of generality that \( y_1 \) is not adjacent to either \( x_3 \) or \( x_4 \). But then \( \{v_1, v_3, x_3, x_4, y_1, z_1, z_3\} \) is an independent set of cardinality 7 in \( \Lambda \), a contradiction. □

The following properties of \( \Lambda \) may be deduced from Lemma 3.

**Lemma 4** \( D_1(\lambda) \) is an independent set of cardinality 4 in \( \Lambda \). Furthermore, \( 8 \leq |D_2(\lambda)| \leq 9 \), \( 4 \leq |D_3(\lambda)| \leq 5 \) and \( D_{>3}(\lambda) = \emptyset \).

**Proof:** It follows by Lemma 3 that \( |D_1(\lambda)| = 4 \). In order to avoid triangles in \( \langle \{\lambda\} \cup D_1(\lambda) \rangle_{\text{red}} \), it follows that \( \langle D_1(\lambda) \rangle_{\text{red}} \) must be edgeless.

Now suppose \( |D_{>2}(\lambda)| \geq 6 \). Since \( t(3, 3) = 6 \), it follows that, in order to avoid a red triangle in \( \Lambda \), the subgraph \( \langle D_{>2}(\lambda) \rangle_{\text{red}} \) must have an independent set, \( Z \) (say), of cardinality 3. But then the set \( D_1(\lambda) \cup Z \) of cardinality 7 is independent in \( \Lambda \). This contradiction shows...
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In this section we show that $t(3, 8) = 22$. We start by producing a $(3, 8, 21)$-colouring in §4.1, showing that $t(3, 8) > 21$. We show in §4.2 that if a $(3, 8, 22)$-colouring exists, each vertex of such a colouring must have red degree 4 or 5. This is followed by a proof in §4.3 that no vertex of a $(3, 8, 22)$-colouring can, in fact, have red degree 5, and hence

that $|D_{>2}(\lambda)| \leq 5$, and hence $|D_2(\lambda)| \geq 8$. Suppose next that $|D_2(\lambda)| \geq 10$. Then $\langle D_2(\lambda) \cup \{w\}\rangle_{\text{red}}$ is bipartite for any vertex $w \in D_{>2}(\lambda)$ by Proposition 4(b), and hence has a partite set, $Z'$ (say), of cardinality at least 6. But then $\{\lambda\} \cup Z'$ is an independent set of cardinality at least 7 in $\Lambda$. This contradiction shows that $|D_2(\lambda)| \leq 9$ and hence that $|D_{>2}(\lambda)| \geq 4$.

If $|D_2(\lambda)| = 9$, then $\langle D_2(\lambda)\rangle_{\text{red}}$ has a partite set of cardinality at least 5 and hence it follows by Lemma 1(f) that $D_{>3}(\lambda) = \emptyset$. Suppose, therefore, that $|D_2(\lambda)| = 8$ and hence that $|D_{>2}(\lambda)| = 5$. Then, in order to avoid red triangles, $\langle D_{>2}(\lambda)\rangle_{\text{red}}$ must be a 5-cycle. However, if any vertex of this 5-cycle is in $\langle D_{>3}(\lambda)\rangle_{\text{red}}$, then that vertex will have degree 2 in $\Lambda$, contradicting the result of Lemma 2. This contradiction shows that $D_{>3}(\lambda)$ must be empty.

We may now prove our main result of this section.

**Theorem 1** $t(3, 7) = 18$.

**Proof:** It follows by Lemma 4 that there are two cases to consider.

*Case i:* $|D_1(\lambda)| = 4$, $|D_2(\lambda)| = 9$ and $|D_3(\lambda)| = 4$. This case may be proven to be impossible by following the exact same arguments as in Case 1 of the proof that $s(3, 7) \leq 18$ in [4], because in these arguments no irredundant set of cardinality 7 is ever avoided which is not also an independent set of cardinality 7.

*Case ii:* $|D_1(\lambda)| = 4$, $|D_2(\lambda)| = 8$ and $|D_3(\lambda)| = 5$. This case may be proven to be impossible by following the exact same arguments as in Cases 2 and 3 of the proof that $s(3, 7) \leq 18$ in [4] for the same reason as cited above.

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Figure 3.2: Part of the $(3, 7, 18)$-colouring $(\Lambda, \overline{\Lambda})$ if $\Delta(\Lambda) = 3$. 

that $|D_{>2}(\lambda)| \leq 5$, and hence $|D_2(\lambda)| \geq 8$. Suppose next that $|D_2(\lambda)| \geq 10$. Then $\langle D_2(\lambda) \cup \{w\}\rangle_{\text{red}}$ is bipartite for any vertex $w \in D_{>2}(\lambda)$ by Proposition 4(b), and hence has a partite set, $Z'$ (say), of cardinality at least 6. But then $\{\lambda\} \cup Z'$ is an independent set of cardinality at least 7 in $\Lambda$. This contradiction shows that $|D_2(\lambda)| \leq 9$ and hence that $|D_{>2}(\lambda)| \geq 4$.

If $|D_2(\lambda)| = 9$, then $\langle D_2(\lambda)\rangle_{\text{red}}$ has a partite set of cardinality at least 5 and hence it follows by Lemma 1(f) that $D_{>3}(\lambda) = \emptyset$. Suppose, therefore, that $|D_2(\lambda)| = 8$ and hence that $|D_{>2}(\lambda)| = 5$. Then, in order to avoid red triangles, $\langle D_{>2}(\lambda)\rangle_{\text{red}}$ must be a 5-cycle. However, if any vertex of this 5-cycle is in $\langle D_{>3}(\lambda)\rangle_{\text{red}}$, then that vertex will have degree 2 in $\Lambda$, contradicting the result of Lemma 2. This contradiction shows that $D_{>3}(\lambda)$ must be empty.

We may now prove our main result of this section.

**Theorem 1** $t(3, 7) = 18$.

**Proof:** It follows by Lemma 4 that there are two cases to consider.

*Case i:* $|D_1(\lambda)| = 4$, $|D_2(\lambda)| = 9$ and $|D_3(\lambda)| = 4$. This case may be proven to be impossible by following the exact same arguments as in Case 1 of the proof that $s(3, 7) \leq 18$ in [4], because in these arguments no irredundant set of cardinality 7 is ever avoided which is not also an independent set of cardinality 7.

*Case ii:* $|D_1(\lambda)| = 4$, $|D_2(\lambda)| = 8$ and $|D_3(\lambda)| = 5$. This case may be proven to be impossible by following the exact same arguments as in Cases 2 and 3 of the proof that $s(3, 7) \leq 18$ in [4] for the same reason as cited above.

4 The Ramsey number $t(3, 8)$

In this section we show that $t(3, 8) = 22$. We start by producing a $(3, 8, 21)$-colouring in §4.1, showing that $t(3, 8) > 21$. We show in §4.2 that if a $(3, 8, 22)$-colouring exists, each vertex of such a colouring must have red degree 4 or 5. This is followed by a proof in §4.3 that no vertex of a $(3, 8, 22)$-colouring can, in fact, have red degree 5, and hence
that the red subgraph of such a colouring must be 4-regular. It is finally shown in §4.4, by considering a number of exhaustive cases, that the assumption of the existence of a 4-regular red subgraph of a \((3, 8, 22)\)-colouring leads to a contradiction in each case, implying that \(t(3, 8) \leq 22\).

4.1 The lower bound \(t(3, 8) > 21\)

Consider the graph \(\Psi\) of order 21 in Figure 4.1. It is easily verifiable that \(\Psi\) is triangle-free and has no 6-cycle in which all three diagonals are absent. It therefore follows by Proposition 2 that \(\Psi\) has no irredundant set of cardinality 3. Furthermore, \(\Psi\) has no independent set of order 8, so that the red-blue edge colouring \((\Psi, \overline{\Psi})\) is a \((3, 8, 21)\)-colouring.

![Figure 4.1: The red subgraph \(\Psi\) of a \((3, 8, 21)\)-colouring \((\Psi, \overline{\Psi})\).](image)

4.2 Properties of any \((3, 8, 22)\)-colouring

Suppose there exists a \((3, 8, 22)\)-colouring. Let \(\Xi\) and \(\overline{\Xi}\) be respectively the red and blue subgraphs of this colouring, and denote the minimum and maximum degrees of \(\Xi\) by \(\delta(\Xi)\) and \(\Delta(\Xi)\), respectively. Suppose \(\xi\) is a vertex of red degree \(\Delta(\Xi)\) in this colouring. Then it follows by Proposition 3 that

\[
4 \leq \delta(\Xi) \leq \Delta(\Xi) \leq 7.
\]

(4.1)

The colouring \(\Xi\) has the following properties.

**Lemma 5** \(D_1(\xi)\) is an independent set of \(\Xi\), \(|D_2(\xi)| \leq 11\) and \(|D_{>2}(\xi)| < t(3, 8 - \Delta(\Xi))\).

**Proof:** \(D_1(\xi)\) is an independent set in \(\Xi\), because it induces a clique in \(\overline{\Xi}\) in order to avoid triangles in \(\langle\{\xi\} \cup D_1(\xi)\rangle_{\text{red}}\), which are prohibited by Proposition 2. Furthermore, \(|D_1| = \Delta(\Xi)\).

Suppose \(|D_2(\xi)| \geq 12\) and let \(u \in D_{>2}(\xi)\). Then it follows, by Proposition 4(b), that \(X = D_2(\xi) \cup \{u\}\) induces a bipartite subgraph (of order at least 13) of \(\Xi\). One of the
partite sets, $A$ say, of this bipartite subgraph has cardinality at least 7.  But then the set $A \cup \{\xi\}$ is an independent set of cardinality at least 8 in $\Xi$, a contradiction.  Hence $|D_2(\xi)| \leq 11$.

Now suppose $|D_{>2}(\xi)| \geq t(3,8-\Delta(\Xi))$.  Then $\langle D_{>2}(\xi) \rangle_{\text{red}}$ possesses an independent set $I$ of cardinality $8-\Delta(\Xi)$.  But then $D_1(\xi) \cup I$ is an independent set of cardinality 8 in $\Xi$, a contradiction.  Hence $|D_{>2}(\xi)| < t(3,8-\Delta(\Xi))$.

It is possible to improve the bounds on $\Delta(\Xi)$ in (4.1).

**Lemma 6**  $4 \leq \delta(\Xi) \leq \Delta(\Xi) \leq 5$.

**Proof:**  Suppose $\Delta(\Xi) = 7$.  Then $|D_1(\xi)| = 7$, $|D_2(\xi)| \leq 11$ and $|D_{>2}(\xi)| < t(3,1) = 1$ by Lemma 5, and so $|D_1(\xi)| + |D_2(\xi)| + |D_{>2}(\xi)| < 7 + 11 + 1 = 19$, a contradiction.

Next suppose $\Delta(\Xi) = 6$.  Then $|D_1(\xi)| = 6$, $|D_2(\xi)| \leq 11$ and $|D_{>2}(\xi)| < t(3,2) = 3$ by Lemma 5, and so $|D_1(\xi)| + |D_2(\xi)| + |D_{>2}(\xi)| < 6 + 11 + 3 = 20$, again a contradiction.  $\blacksquare$

4.3  $\Delta(\Xi) \neq 5$

Suppose $\Delta(\Xi) = 5$.  Then it follows by Lemma 5 that $|D_1(\xi)| = 5$, $|D_2(\xi)| \leq 11$ and $|D_{>2}(\xi)| \leq 5$.  But since $|D_1(\xi)| + |D_2(\xi)| + |D_{>2}(\xi)| = 21$, it must hold that $|D_2(\xi)| = 11$ and $|D_{>2}(\xi)| = 5$.

The subgraph $\langle D_2(\xi) \rangle_{\text{red}}$ of $\Xi$ is bipartite by Proposition 4(b).  Suppose $\langle D_2(\xi) \rangle_{\text{red}}$ comprises $c$ components and denote the partite sets of $\langle D_2(\xi) \rangle_{\text{red}}$ by $X = \cup_{\ell=1}^{c} X_\ell$ and $Y = \cup_{\ell=1}^{c} Y_\ell$.  Then we may assume that $|X| = 6$ and $|Y| = 5$, and that $\langle D_2(\xi) \rangle_{\text{red}}$ has exactly one component, $(X_c,Y_c)$ (say), for which $|X_c| = |Y_c| + 1$, while all other components are balanced (that is, $|X_\ell| = |Y_\ell|$ for all $\ell = 1, \ldots, c-1$).  Note that $|D_{>3}(\xi)| = \emptyset$ by Lemma 1(f).  Hence $\langle D_3(\xi) \rangle_{\text{red}}$ must be a 5-cycle, in order to avoid triangles in $\Xi$ and $\Xi$.  Furthermore,

$$|X_c| \geq 3,$$

(4.2)

for if $|X_c| \leq 2$, then it would follow by Lemma 1(b) that at least three vertices in $D_3(\xi)$ send red edges to some vertex in $X_c$, thus forming a triangle in $\Xi$.  Since $\langle D_3(\xi) \rangle_{\text{blue}}$ contains an odd cycle by Lemma 1(d), we conclude that $\langle D_2(\xi) \rangle_{\text{red}}$ must have at least two balanced components (that is, at least three components in total).  Note that, since $\langle D_3 \rangle_{\text{red}}$ is a 5-cycle, only one pair of red edges can go to a $(1,1)$-component of $\langle D_3 \rangle_{\text{red}}$ (for otherwise a triangle will be forced in $\Xi$).  In view of these restrictions and inequality (4.2), it therefore necessarily follows that

$$|X_1| = |Y_1| = 1, \quad |X_2| = |Y_2| = |Y_3| = 2 \quad \text{and} \quad |X_3| = 3.$$  

(4.3)

Note that the component $(X_2,Y_2)$ must receive four pairs of red edges from $D_3(\xi)$, since the component $(X_1,Y_1)$ can only receive one such pair of edges.

We show that the cardinalities in (4.3) lead to a contradiction, and hence that our supposition that $\Delta(\Xi) = 5$ was wrong.  Denote the 5-cycle of $\langle D_3(\xi) \rangle_{\text{blue}}$ by $v_1 v_2 v_3 v_4 v_5 v_1$.  In order to avoid triangles in $\Xi$, it follows by Lemma 1(a) that the pairs of red edges sent by the five edges of $\langle D_3(\xi) \rangle_{\text{blue}}$ to $D_2(\xi)$ must occur in alternating fashion between the
partite sets $X_2$ and $Y_2$, as shown in Figure 4.2. But then the edge $x_2y_2$ must be blue in order to avoid a red 6-cycle $x_2y_2v_4v_2v_5v_3$ with blue diagonals; notice that the edges $x_2v_2$ and $y_2v_5$ are blue by Lemma 1(a). Similarly, the edge $x_2y_1$ must be blue in order to avoid a red 6-cycle $x_2y_1v_2v_4v_1v_3$ with blue diagonals. But then $x_2$ is isolated in the component $(X_2, Y_2)$ of $\langle D_2(\xi) \rangle_{\text{red}}$, a contradiction.

4.4 $\Xi$ is not 4-regular

If $\Xi$ is 4-regular, then it follows by Lemma 5 that $|D_2(\xi)| \leq 11$ and $|D_{>2}(\xi)| < t(3, 4) = 9$. Therefore, if $\Xi$ is 4-regular, then there are five cases to consider, as outlined in Table 4.1. Note that $|Y| \leq |X| \leq 6$ in order to avoid an independent set $\{\xi\} \cup X$ of cardinality 8 in $\Xi$; hence the five cases in the table.

| Case | $|D_1(\xi)|$ | $|D_2(\xi)|$ | $|D_{>3}(\xi)|$ | $|X|$ | $|Y|$ | Considered |
|------|-------------|-------------|----------------|-----|-----|-----------|
| I    | 4           | 11          | 6              | 6   | 5   | in Lemma 7 |
| IIa  | 4           | 10          | 7              | 5   | 5   | in Lemma 8 |
| IIb  | 4           | 10          | 7              | 6   | 4   | in Lemma 9 |
| IIIa | 4           | 9           | 8              | 5   | 4   | in Lemma 10 |
| IIIb | 4           | 9           | 8              | 6   | 3   | in Lemma 10 |

Table 4.1: Five cases to consider if $\Xi$ is 4-regular.

**Lemma 7** Case I in Table 4.1 is impossible.

**Proof:** In Case I in Table 4.1 it follows by Lemma 1(f) that $D_{>3}(\xi) = \emptyset$ and hence that $|D_3(\xi)| = 6$. Furthermore, since $6 = |X| > |Y| = 5$, there is exactly one component of the bipartite graph $\langle D_2(\xi) \rangle_{\text{red}}$, $(X_c, Y_c)$ (say), for which $|X_c| > |Y_c|$, while all other components have partite sets of equal cardinalities. It follows by Lemma 1(b) that $|X_c| \neq 1$, since $\Delta(\Xi) = 4$. Hence it follows by Corollary 2(b) that $|X_c| \geq |D_3(\xi)| = 6$. But since $|X_c| \leq |X| = 6$, we must have that $|X_c| = 6$, so that $\langle D_2(\xi) \rangle_{\text{red}}$ has only one component. Furthermore, since $r(3, 3) = 6$ and since $\langle D_3(\xi) \rangle_{\text{red}}$ contains no triangle, it follows that $\langle D_3(\xi) \rangle_{\text{blue}}$ contains a triangle, contradicting Lemma 1(d). $\blacksquare$
Lemma 8 Case IIa in Table 4.1 is impossible.

Proof: Since \( \langle D_{\geq 3}(\xi) \rangle_{\text{red}} \) contains no independent set of order 4, it must be the red subgraph of a \( t(3,4) \)-avoidance graph, i.e. one of the graphs \( E_1 - E_8 \) in [3, Figure 1(a)–(h)].

Note that there are only eight avoidance graphs of order 7 for \( s(3,4) [3, \text{Table 3}] \), and since \( \beta(G) \leq IR(G) \) for any graph \( G \), it follows that this set of avoidance \( s(3,4) \)-graphs is also a complete set of avoidance \( t(3,4) \)-graphs of order 7. However, since \( \Delta(E_i) \leq 3 \) for all \( i = 1, \ldots, 8 \) (because \( \Xi \) is 4-regular) and since only vertices of \( E_i \) with degree 4 can be in \( \langle D_{\geq 3}(\xi) \rangle_{\text{red}} \), we have that \( D_{\geq 3} = \emptyset \).

Furthermore, it follows by Lemma 1(d) that the pairs of red edges sent by the edges of a triangle in \( \langle D_3(\xi) \rangle_{\text{blue}} \) to \( D_2(\xi) \) must go to at least two different components of \( \langle D_2(\xi) \rangle_{\text{red}} \). This implies that each triangle in \( \langle D_3(\xi) \rangle_{\text{blue}} \) sends at least five red edges to \( D_2(\xi) \). Note that if such a triangle sends exactly five red edges to \( D_2(\xi) \), then its vertices send respectively 1, 2 and 2 red edges to \( D_2(\xi) \) as they are traversed around the triangle. The complements of each of the avoidance graphs \( E_3, E_4, E_7 \) and \( E_8 \) has a triangle violating the above condition. Therefore \( \langle D_3(\xi) \rangle_{\text{red}} \) must be isomorphic to \( E_1, E_2, E_5 \) or \( E_6 \), shown in Figure 4.3.

![Figure 4.3](image-url)  
Figure 4.3: In case IIa of Table 4.1 \( \langle D_3(\xi) \rangle_{\text{red}} \) must be isomorphic to \( E_1, E_2, E_5 \) or \( E_6 \).

Let \( A \) and \( B \) be two disjoint subsets of the vertex set of \( \Xi \). Then we use the notion \( E^r(A, B) \) to denote the number of edges of \( \Xi \) joining vertices in \( A \) with vertices in \( B \), while \( E^r(A) \) denotes the number of edges of \( \Xi \) joining two vertices of \( A \). Since the sum of the vertex degrees in \( \Xi \) over \( D_3(\xi) \) is

\[
7 \times 4 = E^r(D_2(\xi), D_3(\xi)) + 2E^r(D_3(\xi)), \tag{4.4}
\]

and

\[
E^r(\Xi) = E^r(\{\xi\}, D_1(\xi)) + E^r(D_1(\xi), D_2(\xi)) + E^r(D_2(\xi), D_3(\xi)) + E^r(D_3(\xi)) = 4 + 12 + E^r(D_2(\xi)) + (28 - 2E^r(D_3(\xi))) + E^r(D_3(\xi)) = 44,
\]

it follows that

\[
E^r(D_2(\xi)) = E^r(D_3(\xi)). \tag{4.5}
\]

Furthermore, we observe from Figure 4.3 that \( E^r(D_3(\xi)) = 6, 7 \) or 8, and hence we have three subcases to consider.

Subcase \( i: E^r(D_3(\xi)) = 6 \). In this subcase it follows by (4.4) that \( E^r(D_2(\xi), D_3(\xi)) = 28 - 12 = 16 \). No vertex \( x \in D_2(\xi) \) can receive three red edges from \( D_3(\xi) \), for otherwise \( x \)
would be isolated in \(\langle D_2(\xi)\rangle_{\text{red}}\), which is impossible, since all components of \(\langle D_2(\xi)\rangle_{\text{red}}\) are balanced in Case IIa. It follows by Lemma 1(e) that each vertex in \(D_2(\xi)\) receiving two red edges from \(D_3(\xi)\) must be in a \((1,1)\)-component of \(\langle D_2(\xi)\rangle_{\text{red}}\). Furthermore, if \(\langle D_2(\xi)\rangle_{\text{red}}\) has more than three \((1,1)\)-components, then the maximum number of edges in \(\langle D_2(\xi)\rangle_{\text{red}}\) is 5, contradicting the fact that \(E'_r(D_3(\xi)) = 6\). Therefore, to accommodate all 16 red edges from \(D_3(\xi)\), it follows that \(\langle D_3(\xi)\rangle_{\text{red}}\) necessarily comprises three \((1,1)\)-components and one \((2,2)\)-component, in which case each of the six vertices in the \((1,1)\)-components of \(\langle D_2(\xi)\rangle_{\text{red}}\) receives exactly two red edges from \(D_3(\xi)\).

Let \(\langle \{x_1, y_1\}\rangle_{\text{red}}\) be a \((1,1)\)-component of \(\langle D_2(\xi)\rangle_{\text{red}}\), and let \(N(x_1) \cap D_3(\xi) = \{w_1, w_2\}\) and \(N(y_1) \cap D_3(\xi) = \{w_3, w_4\}\). Note that \(w_1, w_2, w_3, w_4\) are all distinct in order to avoid triangles in \(\Xi\). It follows by Lemma 1(c) that the blue edge \(w_1w_2\) must send a pair of red edges, \(w_1x_2\) and \(w_2y_2\) (say), to some component of \(\langle D_2(\xi)\rangle_{\text{red}}\). Furthermore, it follows by Proposition 4(a) and the fact that \(x_1\) has degree 4 that \(x_1, x_2\) and \(y_2\) must all have one common neighbour, \(v\) (say), in \(D_1(\xi)\). Note that since \(x_2\) and \(y_2\) are both adjacent to \(v\), both \(x_2\) and \(y_2\) must be in the \((2,2)\)-component of \(\langle D_2(\xi)\rangle_{\text{red}}\) in which case \(x_2y_2\) must be blue (in order to avoid a red triangle in \(\Xi\)). Similarly, since the \((2,2)\)-component of \(\langle D_2(\xi)\rangle_{\text{red}}\) contains exactly three red edges, the blue edge \(w_3w_4\) must also send red edges to \(x_2\) and \(y_2\), but this contradicts Lemma 1(e).

**Subcase ii:** \(E'(D_3(\xi)) = 7\). In this subcase it follows by (4.4) that \(E'(D_2(\xi), D_3(\xi)) = 28 - 14 = 14\). Again, since no vertex in \(D_2(\xi)\) can receive three red edges from \(D_3(\xi)\), there must be at least four vertices, \(x_1, y_1, x_3, y_3\) (say), in \(D_2(\xi)\) which each receives two red edges from \(D_3(\xi)\). Hence there are at least two \((1,1)\)-components in \(\langle D_2(\xi)\rangle_{\text{red}}\). Since \(E'(D_2(\xi)) = 7\), there can be at most three \((1,1)\)-components in \(\langle D_2(\xi)\rangle_{\text{red}}\). Without loss of generality, let \(x_1\) and \(y_1\) be in the same \((1,1)\)-component of \(\langle D_2(\xi)\rangle_{\text{red}}\), and let \(w_1, w_2, w_3, w_4\) be the neighbours of \(x_1\) and \(y_1\) in \(D_3(\xi)\). As in subcase i, the blue edge \(w_1w_2\) must send a pair of red edges, \(w_1x_2\) and \(w_2y_2\) (say), to some component of \(\langle D_2(\xi)\rangle_{\text{red}}\). Also, \(x_1, x_2\) and \(y_2\) must have a common neighbour, \(v\) (say), in \(D_1(\xi)\). Since \(D_3(\xi)\) contains exactly seven vertices, it follows by the pigeonhole principle that at least one of \(x_3\) or \(y_3\) must be joined by means of a red edge to one of \(w_1, w_2, w_3, w_4\). We may therefore assume that \(x_3w_1\) is red. But then \(x_1, x_2\) and \(x_3\) must have a common neighbour in \(D_1(\xi)\), which is impossible, since \(x_1\) and \(v\) both already have red degree 4.

**Subcase iii:** \(E'(D_3(\xi)) = 8\) and hence \(\langle D_3(\xi)\rangle_{\text{red}} \cong E_6\). In this subcase it follows by (4.4) that \(E'(D_2(\xi), D_3(\xi)) = 28 - 16 = 12\). There are at least two vertices, \(x_1\) and \(x_2\) (say), in \(D_2(\xi)\) which each receives two red edges from \(D_3(\xi)\). Label the vertices of \(D_3(\xi)\) as in Figure 4.4.

We consider to which pairs of vertices in \(D_3(\xi)\) the vertices \(x_1\) and \(x_2\) can send red edges. Note that \(x_1\) cannot send red edges to pairs of vertices of the form \(\{w_4, w_1\}\) or \(\{w_7, w_j\}\), where \(w_4w_1\) and \(w_7w_j\) are blue edges, since \(w_4\) or \(w_7\) will be saturated in terms of its red degree, therefore either contradicting Lemma 1(c) or forming a red triangle in \(\Xi\). Hence the vertex \(x_1\) must send red edges to two nonadjacent vertices in \(\{w_1, w_2, w_3, w_5, w_6\}\). However, \(x_1\) may not send red edges to the following pairs of vertices:

- \(\{w_2, w_6\}\), for otherwise \(x_1w_2w_3w_4w_5w_6x_1\) would form a red 6-cycle with blue diagonals,
- \(\{w_2, w_5\}\), by symmetry, for the same reason as above,
- \(\{w_1, w_5\}\), for otherwise \(x_1w_1w_2w_3w_4w_5w_6x_1\) would form a red 6-cycle with blue diagonals,
- \(\{w_3, w_6\}\), by symmetry, for the same reason as above.
The Ramsey number $t(3,8)$

Proof: Consider the case where $x_1$ sends red edges to $\{w_1, w_6\}$ and $x_2$ sends red edges to $\{w_1, w_3\}$. Suppose, without loss of generality, that $x_2$ sends red edges to $\{w_1, w_6\}$. Then $w_1$ is saturated in terms of its red degree, which means that the blue edge $w_1w_3$ cannot send a pair of red edges to $D_2(\xi)$ as dictated by Lemma 1. We conclude, without loss of generality, that $x_1$ must send red edges to $\{w_1, w_6\}$, while $x_2$ sends red edges to $\{w_3, w_5\}$ (see Figure 4.4).

Note that $x_1x_2$ must be blue, for otherwise $x_1x_2w_5w_4w_7w_1x_1$ would form a red 6-cycle with blue diagonals. Therefore $x_1$ and $x_2$ are in different $(1,1)$-components of $\langle D_2(\xi) \rangle_{\text{red}}$, and the blue edges $w_1w_6$ and $w_3w_5$ must send pairs of red edges to the remaining components of $\langle D_2(\xi) \rangle_{\text{red}}$. Suppose, without loss of generality, that $w_1, w_3$ and $w_6$ send red edges to $z_1, z_2 \in \xi$ and $z_3 \in Y$, as shown in Figure 4.4. Then we can select two vertices, $v_1$ and $v_2$ (say), from $Y$ which are not joined by means of red edges to the saturated vertices $\{w_3, w_5, w_6\}$, which means that $\langle \{\xi, v_1, v_2, x'_1, x'_2, w_1, w_5, w_6\} \rangle_{\text{blue}}$ is a clique of order 8 in $\Xi$, a contradiction.

Lemma 9 Case IIIb in Table 4.1 is impossible.

Proof: In this case there are two possibilities to consider, namely where $\langle D_2(\xi) \rangle_{\text{red}}$ has two unbalanced components, or where $\langle D_2(\xi) \rangle_{\text{red}}$ has one unbalanced component. We consider the first of these possibilities first, where $(X_i, Y_i)$ are components in question, with $|X_i| = |Y_i| + 1$ for $i = 1, 2$. It follows by Lemma 1(f) that $D_{\geq 3}(\xi) = \emptyset$. Every vertex $w \in D_3(\xi)$ must send a red edge to $X_1 \cup X_2$ in order to avoid the clique of order 8 in $\Xi$ induced by $\{\xi, w\} \cup X_1 \cup X_2$ together with the partite sets of the balanced components of $\langle D_2(\xi) \rangle_{\text{red}}$ which do not receive red edges from $w$. Let $t$ denote the number of red edges incident with vertices in $X_1 \cup X_2$. Then

\[
t = E'(X_1 \cup X_2, D_3(\xi)) + E'(X_1 \cup X_2, Y_1 \cup Y_2) + E'(X_1 \cup X_2, D_1(\xi)) \\
\geq 7 + 2|X_1| - 2 + 2|X_2| - 2 + |X_1| + |X_2| \\
= 3 + 3|X_1| + 3|X_2|.
\]

Let $\epsilon = t - (3 + 3|X_1| + 3|X_2|)$. Then $3 + 3|X_1| + 3|X_2| + \epsilon \leq 4|X_1| + 4|X_2|$ and so

\[
|X_1| + |X_2| \geq 3 + \epsilon.
\] (4.6)
We show that there is a pair of vertices in $D_3(\xi)$ which have a common neighbour $x_1 \in X_i$ such that $|X_i| \geq 2$, for some $i \in \{1, 2\}$. Since $|X_1| + |X_2| \geq 3$, we have that $|X_1| \geq 2$ or $|X_2| \geq 2$. Assume, without loss of generality, that $|X_1| \geq 2$. Suppose every vertex in $X_1$ sends at most one red edge to $D_3(\xi)$. Each of the vertices in $D_3(\xi) - N^r(X_1)$ must send a red edge to $X_2$. If $|X_2| = 1$, then $\{\xi\} \cup N^r(X_2) \cup X_1$ is an independent set of cardinality at least 8 in $\Xi$, a contradiction. Thus, $|X_2| \geq 2$. Since $7 = |D_3(\xi)| \geq |X_1 \cup X_2|$, there must be two vertices, $v_1$ and $v_2$ (say), in $D_3(\xi)$ which send red edges to a vertex $x_2 \in X_2$. Assume therefore, without loss of generality, that $x_1 \in X_2$, that $x_1v_1$ and $x_1v_2$ are red, and that $|X_2| \geq 2$.

Since $|X_2| \geq 2$, there must be a red path $x_1 y x_2$ of length 2 with $x_1, x_2 \in X_2$ and $y \in Y_2$. But then it follows by Proposition 4(a) that $x_1$ and $x_2$ must have a common neighbour, $w$ (say), in $D_1(\xi)$, thus saturating the vertex $x_1$. The vertices $v_1$ and $v_2$ can send at most one more red edge to $D_2(\xi) \setminus X_2$, for otherwise $w$ or $x_1$ will be oversaturated in order to accommodate the common neighbours in $D_1(\xi)$ of the endpoints of the red paths of order 3 formed in $D_2(\xi) \cup D_3(\xi)$, as necessitated by Proposition 4(a). However, if this additional red edge does not go to $X_1$, or if $v_1$ and $v_2$ do not send additional red edges to $D_3(\xi)$, then a clique of order 8 is induced in $\Xi$ by the vertex subset $\{\xi, v_1, v_2\} \cup X_1 \cup Y_2$ together with the partite sets of the balanced components of $(D_2(\xi))_{\text{red}}$ that do not receive a red edge from either $v_1$ or $v_2$.

Assume therefore, without loss of generality, that $v_2 x_3$ is red, for some $x_3 \in X_1$. Then $wx_3$ is red by Proposition 4(a). Note that $\epsilon \geq 1$ since $v_2$ now sends two edges to $X_1 \cup X_2$. We consider, as subcases, the possible cardinalities of $|X_1|$ and $|X_2|$. Note that if $|X_1| \geq 2$, then $\epsilon \geq 2$ since $x_3$ is then part of a red path of order 3 whose endpoints have a common neighbour in $D_1(\xi)$ other than $w$, by Proposition 4(a).

Before continuing, we note the following two useful observations.

**Observation 1** Let $S = \{p_1, p_2, p_3, p_4, p_5\}$. If $(S)_{\text{red}} \subseteq (D_2(\xi))_{\text{red}}$ is a path $p_1 p_2 p_3 p_4 p_5$ of order 5, then the pairs of vertices $\{p_1, p_3\}$, $\{p_2, p_4\}$ and $\{p_3, p_5\}$ all have distinct common neighbours in $D_1(\xi)$.

**Proof:** It follows by Proposition 4(a) that all three pairs of vertices must have common neighbours in $D_1(\xi)$. To avoid triangles in $\Xi$, only the pairs of vertices $\{p_1, p_3\}$ and $\{p_3, p_5\}$ can possibly share a common neighbour in $D_1(\xi)$. Suppose $p_1$, $p_3$ and $p_5$ all have the same common neighbour, $v_1$ (say), and that $p_2$ and $p_4$ have the same common neighbour, $v_2$ (say), in $D_1(\xi)$. Then $v_1 p_1 p_2 v_2 p_4 p_5 v_1$ is a red 6-cycle with blue diagonals, a contradiction.

**Observation 2** If $y \in Y$ and $|X_i| = |Y_i| + 1$ for $i = 1, 2$, then $E^r(\{y\}, X) \leq 2$.

**Proof:** Suppose, to the contrary, that there is a vertex $y_2 \in Y$ which sends red edges to $u_1$, $u_2$, $u_3 \in X$. Then $u_1$, $u_2$ and $u_3$ must have a common neighbour, $w_2 \neq w$ (say), in $D_1(\xi)$ by Corollary 1. But then $(\{w, w_2\} \cup Y \cup \{v_1, v_2\})_{\text{blue}}$ is a clique of order 8 in $\Xi$, a contradiction.

From the above observations it is easy to see that $(D_2(\xi))_{\text{red}}$ cannot contain a $(4, 3)$-component (or larger): It follows by Observation 2 that $(X_i \cup Y_i)_{\text{red}}$ is either a path of order 7 or else $X_i$ contains a vertex $x$ which sends three red edges to $Y_i$. In both cases all the common neighbours cannot be accommodated (either because $x$ is over-saturated
in terms of its red degree or there are not enough vertices in \( D_1(\xi) \), as may be seen by applying Observation 1 on the three subpaths of order 5 of \( p_1, \ldots, p_7 \) starting with \( p_1, p_2 \) and \( p_3 \), respectively. We therefore complete the proof of the lemma by considering two subcases.

**Subcase i:** \( |X_2| = 2 \). Since \( \epsilon \geq 1 \), it follows by (4.6) that \( |X_1| + |X_2| \geq 4 \) and hence that \( |X_1| \geq 2 \). But then \( \epsilon \geq 2 \) by the remark above, implying that \( |X_1| \geq 3 \) by (4.6). Therefore, \( |X_1| = 3 \), and it follows by Observation 2 that \( (X_1 \cup Y_1)_{\text{red}} \) is a path, \( p_1p_2p_3p_4p_5 \) (say), of order 5. But then it follows by Observation 1 that \( \epsilon \geq 3 \), since \( p_3 \) sends two red edges to \( D_1(\xi) \), contradicting the cardinality of \( |X_1| \) in view of (4.6).

**Subcase ii:** \( |X_2| = 3 \). In this subcase \( (X_2 \cup Y_2)_{\text{red}} \) a path of order 5, implying that \( \epsilon \geq 2 \). Hence \( |X_1| + |X_2| \geq 5 \), and so \( |X_1| \geq 2 \). If \( |X_1| = 2 \), then \( \epsilon \geq 3 \), again a contradiction, as above. We conclude that \( |X_1| = 3 \). But if \( (X_1 \cup Y_1)_{\text{red}} \) and \( (X_2 \cup Y_2)_{\text{red}} \) each contains a path of order 5, then the required number of unique common neighbours of the pairs of endpoints of all the subpaths again renders \( \deg(\xi) > 4 \), a contradiction.

The final possibility to consider is where \( (D_2(\xi))_{\text{red}} \) has only one unbalanced component \( (X_1, Y_1) \), with \( |X_1| = |Y_1| + 2 \). Using a similar argument to that used to obtain (4.6), it may be shown that \( |X_1| \geq 4 + \epsilon \). Since \( |X_1| \leq 6 \), we have that \( \epsilon \leq 2 \). Notice that if \( \epsilon = 2 \), then \( (D_2(\xi))_{\text{red}} \) comprises only one component, and \( D_3(\xi) \) sends no red edges to \( Y \). But then \( D_3(\xi) \cup Y \cup \{\xi\} \) induces a clique of order 8 in \( \Xi \) since \( (D_3(\xi))_{\text{blue}} \) must contain a triangle. We therefore conclude that \( \epsilon \leq 1 \).

We complete the proof by showing that the above inequality cannot be satisfied. The subgraph \( (X_1 \cup Y_1)_{\text{red}} \) must either be a connected graph containing a cycle or must contain an induced path of order 5, \( p_1p_2p_3p_4p_5 \) (say). In the former case, \( \epsilon \geq 1 \). In the latter case it follows by Observation 1 that \( p_3 \) sends two red edges to \( D_1(\xi) \) and so again \( \epsilon \geq 1 \). Note that it now follows that \( |X_1| = 5 \). Also, as before, there must be two vertices in \( D_3(\xi) \), \( v_1 \) and \( v_2 \) (say), which send red edges to \( x_1 \in X_1 \). Using a similar argument as in the subcase with two unbalanced components, it follows that \( v_1 \) or \( v_2 \) must send a red edge to \( X_1 - \{x_1\} \) in order to avoid an clique of order 8 in \( \Xi \), implying that \( \epsilon \geq 2 \).}

**Lemma 10** Cases IIIa and IIIb in Table 4.1 are impossible.

**Proof:** In both cases \( |D_{2 \geq 3}(\xi)| = 8 \), and so \( (D_{2 \geq 3}(\xi))_{\text{red}} \) has to be the red subgraph of a (3, 4, 8)-colouring, i.e. one of the graphs \( E_{10} \) or \( E_{11} \) in Figure 4.5, for otherwise a triangle would result in \( \Xi \) or else a clique of order 8 would be induced in \( \Xi \) by the vertices in \( D_1(\xi) \) together with four vertices in \( D_{2 \geq 3}(\xi) \). Since neither \( E_{10} \) nor \( E_{11} \) has a vertex of degree 4, it follows in both cases that, in fact, \( D_{2 \geq 4}(\xi) = \emptyset \).

We first consider the possibility that \( (D_3(\xi))_{\text{red}} \cong E_{10} \) with vertices labelled as in Figure 4.5(a). Each of the vertices \( w_1, w_2, w_3 \) and \( w_4 \) has two neighbours in \( D_2(\xi) \). We show that these neighbours are, in fact, distinct, i.e. that there are eight such neighbours in total. Without loss of generality we only show that the neighbours of \( w_1 \) are distinct from those of \( w_2, w_3 \) and \( w_4 \). Firstly, \( w_1 \) and \( w_4 \) cannot have common neighbourhood in \( D_2(\xi) \), for otherwise a triangle would result in \( \Xi \). Furthermore, if \( w_1 \) and \( w_3 \) have common a neighbour, \( y \) (say), in \( D_2(\xi) \), then the red 6-cycle with blue diagonals \( w_3yw_1w_4x_3x_4w_3 \) results in \( (\Xi, \Xi) \), unless the edge \( x_3y \) is red. However, \( w_3yw_1x_2x_1w_2w_3 \) is similarly a red 6-cycle with blue diagonals in \( (\Xi, \Xi) \), unless the edge \( x_1y \) is red. But \( x_1y \) and \( x_3y \) cannot
The Ramsey number $t(3, 8)$

Figure 4.5: The only two possibilities for the red subgraph of a $(3, 4, 8)$-colouring [3, Table 4].

both be red, for this would over-saturate the vertex $y$. A similar argument shows that $w_1$ and $w_2$ cannot have a common neighbour (in this case the two red 6-cycles with blue diagonals are $w_2w_1x_2x_4w_3w_2$ and $w_2w_1w_4x_3x_1w_2$).

Figure 4.6: A part of $(\Xi, \overline{\Xi})$ in the cases IIIa and IIIb, if $\langle D_3(\xi) \rangle_{\text{red}} \cong E_{11}$.

Define $Z_1 = (N(w_1) \cap D_2(\xi)) \cup \{x\}$ and $Z_i = N(w_i) \cap D_2(\xi)$ for all $i \in \{2, 3, 4\}$, and let $xv_1$ be red without loss of generality, as shown in Figure 4.6. Then each pair of vertices in $Z_i$ must have a common neighbour, $v_i$ (say), in $D_1(\xi)$ by Proposition 4(a), for all $i \in \{1, 2, 3, 4\}$. Note that $N(w_2) \cup \{w_1, v_1\}$, $N(w_3) \cup \{w_4, v_1\}$ and $N(w_4) \cup \{w_3, v_1\}$ each forms a clique of order 6 in $\overline{\Xi}$. Therefore, in order to avoid a clique of order 8 in $\overline{\Xi}$, there must be a red edge between $Z_2$ and $\{v_3, v_4\}$, between $Z_3$ and $\{v_2, v_4\}$, and between $Z_4$ and $\{v_2, v_3\}$. Hence there are three red edges between $Z_2 \cup Z_3 \cup Z_4$ and $\{v_2, v_3, v_4\}$. Since $\Xi$ is 4-regular, there cannot be any red edges between $Z_1$ and $\{v_2, v_3, v_4\}$. But then a clique of order 8 is induced in $\overline{\Xi}$ by the vertices in $Z_1 \cup \{v_2, v_3, v_4, w_3, w_4\}$, a contradiction.

Consider next the possibility that $\langle D_3(\xi) \rangle_{\text{red}} \cong E_{11}$ with vertices labelled $w_1, \ldots, w_8$ as in Figure 4.5(b). In Case IIIa of Table 4.1 the vertices in $\{\xi, w_i, w_j\} \cup X$ will induce a clique of order 8 in $\overline{\Xi}$ if a blue edge $w_iw_j$ in $D_3(\xi)$ sends both its red edges to $Y$. Similarly, the vertices in $\{\xi, w_i, w_j, w_k\} \cup Y$ will induce a clique of order 8 in $\overline{\Xi}$ if a blue triangle $w_iw_jw_k$ in $D_3(\xi)$ sends all its red edges to $X$ in Case IIIa of Table 4.1. Label the vertices in $D_3(\xi)$...
by means of the symbols $x$ and $y$ to indicate whether the vertices send red edges to $X$ or to $Y$, respectively. Thus, in order to avoid a clique of order 8 in $\Xi$, the vertices in $D_3(\xi)$ should be labelled $x$ and $y$ in such a way the endpoints of every blue edge in $D_3(\xi)$ are not both labelled $y$, and such that the vertices of a blue triangle in $D_3(\xi)$ are not all labelled $x$. We show that this is not possible. Since not all vertices in $D_3(\xi)$ can be labelled $x$, some vertex, $w_1$ (say), must be labelled $y$. To avoid labelling both endpoints of blue edges in $D_3(\xi)$ with the symbol $y$, the vertices $w_3$, $w_4$, $w_6$ and $w_7$ must all be labelled $x$. Furthermore, in order to avoid labelling all the vertices of the triangles $\langle\{w_3, w_6, w_8\}\rangle_{\text{blue}}$ and $\langle\{w_2, w_4, w_7\}\rangle_{\text{blue}}$ with the symbol $x$, the vertices $w_2$ and $w_8$ must both be labelled $y$. But then both endpoints of the blue edge $w_2w_8$ in $D_3(\xi)$ are labelled $y$, a contradiction. ■

The following result therefore holds in view of Lemmas 7–10.

**Theorem 2** $t(3, 8) = 22$. 

5 Conclusion

In this paper we established the two new mixed irredundant Ramsey numbers $t(3, 7) = 18$ and $t(3, 8) = 22$. Using (1.1) and the values in Table 1.1 it therefore follows that

$$14 \leq t(4, 5) \leq 25,$$
$$18 \leq t(7, 3) \leq 23,$$
$$18 \leq s(3, 8) \leq 22,$$
$$13 \leq s(4, 5) \leq 25.$$

These are the six smallest unknown Ramsey numbers involving the graph theoretic notion of irredundance, and are certainly worthy of further investigation.

References


REFERENCES


