Enumeration of Self-orthogonal Latin Squares

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Abstract

In this report we consider the problem of enumerating self-orthogonal Latin squares (SOLS) of small orders. We present enumeration tables of unequal SOLS, idempotent SOLS, isomorphism classes of SOLS and isotopy classes of SOLS. The isotopy classes are enumerated by an (almost) exhaustive computerised tree search which generates a representative from each isotopy class, whereafter the autotopism groups of these representatives are used together with results from abstract algebra in order to enumerate unequal SOLS, idempotent SOLS and isomorphism classes of SOLS. Finally, the results are validated by using an alternative computerised search-tree method for all four classes of SOLS.

Keywords: Latin square, Self-orthogonal Latin square, Enumeration, Isomorphism classes, Isotopy classes.

AMS Classification: 05A15, 05B15.

1 Introduction

During the time the work presented in this report was done, we were unaware of a publication by Graham and Roberts [8] concerning the enumeration of self-orthogonal Latin squares (SOLS) of orders $1 \leq n \leq 9$. We only became aware of the paper after we have completed the writing of this report. Although most of our results were found in the Graham and Roberts paper, the results were obtained independently and by utilising different methods and validation techniques. A section of this report does, however, contain new results, and this section have been written up and submitted as a research paper [4]. The results obtained in this report verifies the correctness of the results found by Graham and Roberts, as explained in [4]. It should also be noted that the classes of SOLS defined in this report borrows their names from the Latin square literature (names such as *isomorphism* and *isotopism*), but for the purpose of classifying SOLS they are defined differently for SOLS than for Latin squares in general. Since we never refer to these classes of Latin squares in general, there should be no confusion as to the definitions of these classes. In [4], however, different names are given to these classes in order to avoid ambiguity.

Given a set of n distinct symbols, a *Latin square* of order n is an $n \times n$ array containing each symbol exactly once in every row and every column. In this paper we denote the entry in row i and column j of a Latin square $\boldsymbol{L}$ by $\boldsymbol{L}(i,j)$ and take the n symbols from the set $\mathbb{Z}_n = \{0, \ldots, n-1\}$. We also use $\mathbb{Z}_n$ as index set for the rows and columns of a Latin square. The *transpose* of a Latin square $\boldsymbol{L}$, denoted by $\boldsymbol{L}^T$, is a Latin square for which $\boldsymbol{L}^T(i,j) = \boldsymbol{L}(j,i)$. Two Latin squares $\boldsymbol{L}$ and $\boldsymbol{L}'$ are said to be *orthogonal* if each ordered pair $(\boldsymbol{L}(i,j), \boldsymbol{L}'(i,j))$ is unique among all such pairs for all i and j. If a Latin square $\boldsymbol{L}$ is orthogonal to its transpose, then $\boldsymbol{L}$ is said to be a *self-orthogonal Latin square (SOLS)*. The existence of a SOLS is guaranteed for every order $n \in \mathbb{N}$, except for $n \neq 2, 3, 6$ [3]. For orders 2 and 3 this fact is easily verifiable by exhaustive enumeration, and it is well known that there exists no pair of *mutually orthogonal Latin squares (MOLS)*

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of order 6 [20]. Notable existence and construction results for SOLS appear in Mendelsohn [16], Hedayat [9, 10] and Brayton *et al.* [3].

The purpose of this report is to count the number of different types of SOLS with the aid of a computer and to present the results in tabular form. More specifically, we count the number of unequal SOLS, the number of idempotent SOLS, the number of isomorphism classes of SOLS and the number of isotopy classes of SOLS of orders $4 \leq n \leq 9$. In Section 2 our general approach towards the enumeration of SOLS is discussed in some detail, followed by the enumeration of the isotopy and isomorphism classes in Sections 3 and 4 respectively, and the enumeration of unequal and idempotent SOLS in Section 5. Finally the results are reported in tabular form in Section 6, together with the description of a number of methods used to validate those numerical results. The paper closes with a summary of the work presented and possibilities for future work in Section 7.

2 Our general approach towards enumerating SOLS

We employ an exhaustive depth-first search tree approach with backtracking to enumerate SOLS, enabling us to build up a repository of SOLS in addition to enumerating them. We incorporate various pruning rules in order to limit the size of the tree and hence render the approach tractable.

A *universal* in a Latin square $\boldsymbol{L}$ is a list of n entries taken from $\boldsymbol{L}$, all containing the same element. The enumeration of SOLS is performed by branching on the inclusion of all possible universals of a specific symbol, one at a time, within a given partially completed square, provided that the inclusion of a universal does not destroy the property of self-orthogonality. This process is repeated on every level of the tree for another symbol until either a completed SOLS is found or the inclusion of no further universal is possible.

To maintain self-orthogonality a list of all ordered pairs of symbols is kept, adding the pair $(\boldsymbol{L}(i,j), \boldsymbol{L}^T(i,j))$ to the list once the entry $\boldsymbol{L}(i,j)$ assumes a value during the search process, while avoiding the inclusion of universals which result in listing an ordered pair twice. A *transversal* in a Latin square $\boldsymbol{L}$ is a list of n entries taken from $\boldsymbol{L}$ such that no two entries appear in the same row or column of $\boldsymbol{L}$ and no two entries contain the same symbol. Another way of maintaining self-orthogonality is to note that each universal in $\boldsymbol{L}$ corresponds to a transversal in $\boldsymbol{L}^T$. When an attempt is thus made to include a universal in the square a test may be performed in $\boldsymbol{L}^T$ to ensure that the corresponding elements form at least a partial transversal.

For clarity the pseudocode for an implementation of the method described above, using recursion, is given in Algorithm 1. All enumeration methods described in this paper were implemented in Wolfram's MATHEMATICA 6.0 [21] on an Intel(R) Core(TM) 2 Duo processor with 3.25 GB of RAM, except where otherwise stated, and the trivial cases of SOLS of orders $n = 2, 3$ are omitted. All computing times are reported in the format *dd*:*hh*:*mm*:*ss*.*x* where *dd* represents the number of days, *hh* the number of hours, *mm* the number of minutes, *ss* the number of seconds and *x* the number of tenths of a second.

Algorithm 1 `Recursion`

Input: A partially completed SOLS S of order n, a set $\$$ of completed SOLS and a current element e.
Output: The set $\$$ with addition of all possible completions of S to SOLS of order n.

$U \leftarrow$ the set of all possible universals of element e.
if $U = \emptyset$ and S is a SOLS **then**
  $\$ \leftarrow \$ \cup \{S\}$.
else
  for all $u_i \in U$ **do**
    $S' \leftarrow u_i$ inserted into corresponding entries in S.
    $\$ \leftarrow \texttt{Recursion}(S', \$, e+1)$
  end for
end if
Return $\$$

3 Enumeration of the isotopy classes

Two SOLS $\boldsymbol{L}$ and $\boldsymbol{L}'$ are *isotopic* if two permutations performed on $\boldsymbol{L}$ or $\boldsymbol{L}^T$, one on the rows and columns, and one on the symbol set, results in $\boldsymbol{L}'$. An *isotopism* is an element of the group $S_n \times S_n \times S_2$ acting on a SOLS, where S_n is the symmetric group of order n. We denote an isotopism by $\boldsymbol{p} = (p_r, p_s, p_t)$, where $p_r \in S_n$ is the permutation applied to the rows and columns, $p_s \in S_n$ is the permutation applied to the symbol set and $p_t \in S_2$ permutes the roles of the rows and columns of $\boldsymbol{L}$ (*i.e.* it incorporates matrix transposition). Two different approaches to the enumeration of the isotopy classes were implemented and both utilise the search-tree approach described in Section 2. The first approach completes a set of partially completed SOLS in such a way that the generation of a SOLS from each isotopy class is guaranteed. This method requires a certain degree of isotopy testing on the final set of generated SOLS. The second approach completes an initially empty array by iteratively inserting universals for the symbols $0, 1, \ldots, n-1$ while ensuring that after each universal is inserted the resulting partially completed SOLS cannot be transformed via an isotopism into a lexicographically smaller partially completed SOLS, a method generally referred to as *orderly generation* [7, 14, 18]. This method therefore ensures that a single class-representative from each isotopy class is generated, and no isotopy testing is required.

3.1 Enumeration based on the notion of a skeleton

The diagonal of a SOLS is necessarily a transversal,[1] and the diagonal and any row of a SOLS share a common element, called the *root* of the row. Upon the exclusion of the root $\boldsymbol{L}(j,j)$, the diagonal and row j form the permutation

$$\begin{pmatrix} \boldsymbol{L}(0,0) & \boldsymbol{L}(1,1) & \ldots & \boldsymbol{L}(j-1,j-1) & \boldsymbol{L}(j+1,j+1) & \ldots & \boldsymbol{L}(n-1,n-1) \\ \boldsymbol{L}(j,0) & \boldsymbol{L}(j,1) & \ldots & \boldsymbol{L}(j,j-1) & \boldsymbol{L}(j,j+1) & \ldots & \boldsymbol{L}(j,n-1) \end{pmatrix}$$

whose cycle structure[2] is called the *diagonal cycle structure* of row j and is denoted by $\delta(\boldsymbol{L}, j)$. The diagonal cycle structure for column j is defined similarly and is denoted by $\delta(\boldsymbol{L}^T, j)$. For example, the diagonal cycle structures $\delta(\boldsymbol{S}, i) = x_2^3$ and $\delta(\boldsymbol{S}^T, i) = x_3^2$ for $i = 0, \ldots, 6$, result for the SOLS

$$\boldsymbol{S}_7 = \begin{bmatrix} 0&2&1&4&3&6&5 \\ 3&1&6&0&5&4&2 \\ 4&5&2&6&0&1&3 \\ 5&6&4&3&2&0&1 \\ 6&3&5&1&4&2&0 \\ 1&0&3&2&6&5&4 \\ 2&4&0&5&1&3&6 \end{bmatrix}$$

of order 7. Note that the diagonal cycle structure of any row or column of a SOLS is invariant under an isotopism. The *cycle structure representative* of a diagonal cycle structure $\delta(\boldsymbol{L}, j)$ is the lexicographically smallest permutation that assumes $\delta(\boldsymbol{L}, j)$ as its cycle structure. For instance, the cycle structure representative of $x_2 x_4$ is $\binom{1\,2\,3\,4\,5\,6}{2\,1\,4\,5\,6\,3}$.

A SOLS $\boldsymbol{L}$ is *idempotent* if $\boldsymbol{L}(i,i) = i$ for all $i \in \mathbb{Z}_n$. Our first enumeration approach for non-isotopic SOLS is based on the notion of a *SOLS skeleton*. An $n \times n$ *skeleton* is a partially completed SOLS of order n in which the first row is a cycle structure representative, its diagonal is a transversal in natural order and the only other non-empty entries occur in the first column, which is complete. Since a skeleton is a partially completed SOLS, its first row and column are both derangements with respect to the diagonal, and the

[1] Since each entry on the diagonal of a SOLS forms a pair $(\boldsymbol{L}(i,i), \boldsymbol{L}^T(i,i))$ and since $\boldsymbol{L}(i,i) = \boldsymbol{L}^T(i,i)$, it follows that if any symbol appears more than once on the diagonal, the property of orthogonality is destroyed. Therefore the main diagonal of a SOLS is necessarily a transversal, implying that $\boldsymbol{L}(i,j) \neq \boldsymbol{L}(j,i)$ for $i \neq j$.

[2] We used the standard notation $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ to denote the cycle structure of a permutation of order n, where α_i denotes the number of cycles of length i (see [1, p. 442]).

first row and column are derangements with respect to one another[3] and contain no 2-cycles with respect to each other. These requirements are necessary so as to maintain the property of self-orthogonality when completing a skeleton to form a SOLS. If a SOLS $\boldsymbol{L}$ contains a skeleton (*i.e.* if the first row, the first column and the main diagonal of $\boldsymbol{L}$ form a skeleton), this skeleton is denoted by $\langle \boldsymbol{L} \rangle$. Also, any SOLS that contains a skeleton is idempotent. For instance, the skeleton of $\boldsymbol{S}_7$ is

$$\langle \boldsymbol{S}_7 \rangle = \left[\begin{array}{ccccccc} 0 & 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 1 & & & & & \\ 4 & & 2 & & & & \\ 5 & & & 3 & & & \\ 6 & & & & 4 & & \\ 1 & & & & & 5 & \\ 2 & & & & & & 6 \end{array} \right].$$

A SOLS $\boldsymbol{L}$ is *induced* by a skeleton $\mathcal{K}$ if it is isotopic to a SOLS $\boldsymbol{L}'$ for which $\langle \boldsymbol{L}' \rangle = \mathcal{K}$. If it may be shown that any SOLS is induced by at least one skeleton, then an exhaustive list of skeletons may be presented as starting conditions to the enumeration method described in Section 2.

Theorem 3.1 *Every SOLS is induced by at least one skeleton.*

Proof: The proof is constructive in the sense that we demonstrate an isotopism from any SOLS $\boldsymbol{L}$ to a SOLS $\boldsymbol{L}'$ such that $\boldsymbol{L}'$ contains a skeleton. The first step is to permute all the rows and columns of a SOLS $\boldsymbol{L}$ except the first row and column, such that each cycle in the diagonal cycle structure of the first row has the form

$$\begin{pmatrix} p_1 & p_2 & \cdots & p_{k-1} & p_k \\ p_2 & p_3 & \cdots & p_k & p_1 \end{pmatrix},$$

and that the cycles occur in order of non-decreasing length. Hence the first row has the form of a cycle structure representative. If the symbols are renamed such that the diagonal is in natural order, the first row is a cycle structure representative, and the resulting SOLS contains a skeleton. $\square$

A cycle structure $a = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ is *lexicographically smaller* than a cycle structure $b = x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}$, denoted by $a \prec b$, if $\alpha_i > \beta_i$ for some i and $\alpha_j = \beta_j$ for all $1 \leq j < i$. Our approach is to construct a list of all $n \times n$ skeletons in increasing lexicographic order according to the diagonal cycle structure of the first row. However, not all of these skeletons need be considered, since some will be incapable of producing new SOLS when used as a starting point for the enumeration method described in Section 2.

Consider a permutation on the row, column and diagonal of a skeleton $\mathcal{K}$, which is equivalent to a permutation on the rows and columns of a SOLS with skeleton $\mathcal{K}$. In order for the resulting row, column and diagonal to form a skeleton, an appropriate change in symbols has to be performed in order for the diagonal to be in natural order. If this change in symbols causes the row to form a cycle structure representative, then the resulting skeleton $\mathcal{K}'$ is *isotopic* to $\mathcal{K}$. If this change in symbols causes the column to form a cycle structure representative, then the row and column may be swapped and the resulting skeleton $\mathcal{K}'$ is again isotopic to $\mathcal{K}$. This implies that a SOLS with skeleton $\mathcal{K}$ is isotopic to at least one SOLS with skeleton $\mathcal{K}'$. Thus only non-isotopic skeletons are considered, and this is done by exhaustively considering all isotopes of a skeleton, grouping them together in an isotopism class and repeating the process for a skeleton not yet assigned to a class. One skeleton from each class may then be selected as a starting point for the enumeration method of Section 2. For illustrative purposes all non-isotopic skeletons of orders $n = 4, 5$ and 6 are given in Table 3.1. The non-isotopic skeletons of orders $n = 7, 8$ and 9 may be found in [12].

A small section of the search tree for $n = 7$ is shown in Figure 3.1. It shows three branches for three different universals of the element 4 (after universals for the elements 1, 2 and 3 have already been inserted), how the insertion of two of them destroy the property of self-orthogonality (as underlined) and how one is eventually completed to a SOLS. For this latter branch there is, after the insertion of the universal for the element 4,

[3] The first row and column are derangements with respect to one another if they have no fixed point other than the element they have in common when viewed as a permutation.

$$\begin{bmatrix} 0 & 2 & 3 & 1 \\ 3 & 1 & & \\ 1 & & 2 & \\ 2 & & & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & 1 & 4 & 3 \\ 3 & 1 & & & \\ 4 & & 2 & & \\ 2 & & & 3 & \\ 1 & & & & 4 \end{bmatrix}$$
$$n = 4 \qquad n = 5$$

$$\begin{bmatrix} 0 & 2 & 1 & 4 & 5 & 3 \\ 3 & 1 & & & & \\ 4 & & 2 & & & \\ 5 & & & 3 & & \\ 2 & & & & 4 & \\ 1 & & & & & 5 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 & 4 & 5 & 1 \\ 3 & 1 & & & & \\ 1 & & 2 & & & \\ 5 & & & 3 & & \\ 2 & & & & 4 & \\ 4 & & & & & 5 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 & 4 & 5 & 1 \\ 3 & 1 & & & & \\ 4 & & 2 & & & \\ 5 & & & 3 & & \\ 1 & & & & 4 & \\ 2 & & & & & 5 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & & & & \\ 1 & & 2 & & & \\ 2 & & & 3 & & \\ 3 & & & & 4 & \\ 4 & & & & & 5 \end{bmatrix}$$
$$n = 6$$

Table 3.1: All non-isotopic skeletons of orders $n = 4, 5$ and 6.

only one possible universal for the element 5 and similarly for the element 6. Note that the universals are inserted in the order $1, 2, \ldots, n-1$, while the universal for the element 0 is uniquely defined once all other universals are inserted. Since the only appearance of the element 0 in a skeleton is in the top-left corner, the element 0 initially has, above all other elements, the largest number of possible universals, and is thus not inserted until the very end.

$$\begin{bmatrix} 0 & 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 1 & & & & & 2 \\ 4 & & 2 & & & 1 & 3 \\ 5 & & & 3 & 2 & & 1 \\ 6 & 3 & & 1 & 4 & 2 & \\ 1 & & 3 & 2 & & 5 & \\ 2 & & & & 1 & 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 1 & & & & \mathbf{4} & 2 \\ 4 & & 2 & & & 1 & \underline{3} \\ 5 & \mathbf{4} & & 3 & 2 & & 1 \\ 6 & 3 & & 1 & 4 & 2 & \\ 1 & & 3 & 2 & & 5 & \underline{\mathbf{4}} \\ 2 & & \underline{\mathbf{4}} & & 1 & \underline{3} & 6 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 1 & \mathbf{4} & & & & \underline{2} \\ 4 & & 2 & & & 1 & 3 \\ 5 & & & 3 & 2 & \underline{\mathbf{4}} & 1 \\ 6 & 3 & & 1 & 4 & 2 & \\ 1 & & 3 & \underline{2} & & 5 & \mathbf{4} \\ 2 & \underline{\mathbf{4}} & & & 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 1 & & & & \mathbf{4} & 2 \\ 4 & & 2 & & & 1 & 3 \\ 5 & & \mathbf{4} & 3 & 2 & & 1 \\ 6 & 3 & & 1 & 4 & 2 & \\ 1 & & 3 & 2 & & 5 & \mathbf{4} \\ 2 & \mathbf{4} & & & 1 & 3 & 6 \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} 0 & 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 1 & 6 & 0 & 5 & 4 & 2 \\ 4 & 5 & 2 & 6 & 0 & 1 & 3 \\ 5 & 6 & 4 & 3 & 2 & 0 & 1 \\ 6 & 3 & 5 & 1 & 4 & 2 & 0 \\ 1 & 0 & 3 & 2 & 6 & 5 & 4 \\ 2 & 4 & 0 & 5 & 1 & 3 & 6 \end{bmatrix}$$

Figure 3.1: A small section of the search tree for $n = 7$.

Theorem 3.1 guarantees that no SOLS are omitted during the search, but it does not guarantee that the SOLS eventually enumerated will all be non-isotopic (in pairs). Also, if two skeletons are non-isotopic it does not necessarily mean the SOLS generated from them are non-isotopic. An isotopy test therefore has to be performed on the set of enumerated SOLS so as to rid the set of redundant SOLS.

An efficient method for discarding a few redundant SOLS stems from the realisation that if all skeletons for which the first row has the cycle structure x_2x_4 have been completed to SOLS, any SOLS $\boldsymbol{L}$ found at a later stage for which $\delta(\boldsymbol{L}, j) = x_2x_4$ or $\delta(\boldsymbol{L}^T, j) = x_2x_4$ for any j is isotopic to some SOLS already enumerated. This implies that if, for any enumerated SOLS $\boldsymbol{L}$, $\delta(\boldsymbol{L}, j) \prec \delta(\boldsymbol{L}, 0)$ or $\delta(\boldsymbol{L}^T, j) \prec \delta(\boldsymbol{L}, 0)$ for any j, $\boldsymbol{L}$ may be discarded from the set. The above-mentioned method is useful, but it still does not guarantee that all SOLS in the remaining set are non-isotopic.

Consider two idempotent SOLS $\boldsymbol{L}$ and $\boldsymbol{L}'$ for which we want to test for an isotopism. Since both are idempotent, any change of symbols in $\boldsymbol{L}$ must be followed by a unique permutation on the rows and columns of $\boldsymbol{L}$ in order for the resulting square to be idempotent. Hence an appropriate isotopy test consists of only finding the specific change of symbols in $\boldsymbol{L}$ which will result in $\boldsymbol{L}'$. Consider a permutation on the symbol set of $\boldsymbol{L}$ such that a is replaced by α and b by β, and suppose a, b, c and d form the subsquare in $\boldsymbol{L}$ shown in Figure 3.2 (a), where a and b lie on the diagonal of $\boldsymbol{L}$. Since $\boldsymbol{L}$ is a SOLS it necessarily holds that $a \neq b$ and $c \neq d$. Similarly, let α, β, γ and δ form the subsquare in $\boldsymbol{L}'$ shown in Figure 3.2 (b), where α and β lie on the diagonal of $\boldsymbol{L}'$. It is easy to see that these subsquares remain intact during any isotopism from $\boldsymbol{L}$ to $\boldsymbol{L}'$, and thus that if a is replaced by α and b by β, then c must be replaced by γ and d by δ. If we initially apply a change in name on two symbols, it will immediately imply a change in name on two more symbols, eventually causing a chain reaction resulting in either a partially completed permutation or a contradiction. This process may be repeated until either a complete permutation on the symbol set is found, thus implying an isotopism from $\boldsymbol{L}$ to $\boldsymbol{L}'$, or until all possibilities result in contradictions, implying that $\boldsymbol{L}$ and $\boldsymbol{L}'$ are non-isotopic.

$$\begin{array}{ccc} a & \dots & c \\ \vdots & & \vdots \\ d & \dots & b \end{array} \qquad \begin{array}{ccc} \alpha & \dots & \gamma \\ \vdots & & \vdots \\ \delta & \dots & \beta \end{array}$$

(a) (b)

Figure 3.2: Two subsquares of a SOLS.

We denote the number of non-isotopic skeletons of order n by $\kappa(n)$ and the number of SOLS of order n generated from the $\kappa(n)$ skeletons by $\Gamma(n)$. The number of isotopy classes of SOLS of order n, found by applying the methods described above to the SOLS generated from the skeletons, is denoted by $\Psi(n)$. The results for the enumeration of skeletons and the isotopy classes are given in Table 3.2 together with the required computing times. The depth-first tree search used to enumerate $\Gamma(9)$ was achieved by means of 15 computers running in parallel. Two computers were as previously mentioned in Section 2, two were Intel(R) Pentium(R) D processors with 1.024 GB of RAM each, one was an Intel(R) Pentium IV processor with 512 MB of RAM and the remainder were Intel(R) Core(TM) 2 Duo processors with 1.024 GB of RAM each.

	Skeleton generation		Depth-first Tree Search		Final isotopy testing	
n	$\kappa(n)$	Time	$\Gamma(n)$	Time	$\Psi(n)$	Time
4	1	0.0	1	0.0	1	—
5	1	0.0	1	0.0	1	—
6	4	0.1	0	0.1	0	—
7	20	0.8	8	3.5	4	1.3
8	78	14.6	12	9:25.2	4	2.2
9	595	11:57.7	1561	19:20:11:00.5	175	2:55:35.1

Table 3.2: The number, $\Psi(n)$, of isotopy classes of SOLS of order n together with the required computing times.

A complete set of all $\Psi(n)$ isotopy class representatives of SOLS of order n appear in the appendix for all $4 \leq n \leq 9$.

3.2 An alternative enumeration approach

Our second approach is to initialise with an empty array and use the search-tree approach described in Section 2 together with a number of additional pruning rules in order to generate a single class representative from each isotopy class.

The $(3, 2, 1)$-*conjugate* of a SOLS $\boldsymbol{L}$ for which $\boldsymbol{L}(i, j) = k$, denoted by $\boldsymbol{L}_{(3,2,1)}$, is a Latin square for which $\boldsymbol{L}_{(3,2,1)}(k, j) = i$. Note that if $\boldsymbol{L}$ is idempotent, then so is $\boldsymbol{L}_{(3,2,1)}$. Given a set of n distinct symbols, a $k \times n$

row-scattered Latin rectangle (RSLR) for $k \leq n$ is a Latin square of order n with $n-k$ of its rows removed, in other words a Latin squares with $n-k$ empty rows and k complete rows. A RSLR $\boldsymbol{R}$ is *idempotent* if $\boldsymbol{R}(i,i) = i$ for every row i that is not empty. The $(1,3,2)$-*conjugate* of $\boldsymbol{R}$, denoted by $\boldsymbol{R}_{(1,3,2)}$, is the RSLR resulting from the replacement of each row by its inverse permutation. Two RSLRs $\boldsymbol{R}$ and $\boldsymbol{R}'$ are *pseudo-isotopic* if two permutations performed on $\boldsymbol{R}$ or $\boldsymbol{R}_{(1,3,2)}$, one on the columns and symbol set and one on the rows, results in $\boldsymbol{R}'$. It may be noted that a permutation applied to the rows of an RSLR permutes the compelete rows as well as the empty rows.

Given a $k \times n$ RSLR $\boldsymbol{R}$ and a $k \times n$ RSLR $\boldsymbol{R}'$, let $\ell \leq k$ be the smallest index for which row ℓ of either $\boldsymbol{R}$ or $\boldsymbol{R}'$ is empty. Then $\boldsymbol{R}$ is *lexicographically smaller* than $\boldsymbol{R}'$ if the rows indexed by $0, 1, \ldots, i-1$ for some index $i < \ell$ are equal for both $\boldsymbol{R}$ and $\boldsymbol{R}'$, and row i in $\boldsymbol{R}$ is lexicographically smaller (in a permutation sense) than row i in $\boldsymbol{R}'$. If all rows indexed by $i < \ell$ are equal, then the lexicographic order of $\boldsymbol{R}$ and $\boldsymbol{R}'$ is undecided. The lexicographically smallest SOLS in a single isotopy class is defined to be the *isotopy class leader* of that class.

It may be noted that each universal in a SOLS $\boldsymbol{L}$ corresponds to the row in $\boldsymbol{L}_{(3,2,1)}$ indexed by its symbol and that an isotopism performed on $\boldsymbol{L}$ corresponds to a pseudo-isotopism performed on $\boldsymbol{L}_{(3,2,1)}$. Also, after universals for the symbols $0, 1, \ldots, k-1$ have been inserted into an empty $n \times n$ array, the $(3,2,1)$-conjugate of the partially completed SOLS is a $k \times n$ RSLR. If there exists no pseudo-isotopism which transforms this RSLR into a lexicographically smaller RSLR, we henceforth refer to such a RSLR as a *candidate* with respect to SOLS isotopy class leadership. Our approach is to branch on a partially completed SOLS only if the $(3,2,1)$-conjugate is a candidate, and in this way a single representative from each isotopy class of SOLS is generated.

Consider any non-empty row of a $k \times n$ RSLR $\boldsymbol{R}$ as a permutation

$$r = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ r(0) & r(1) & \cdots & r(n-1) \end{pmatrix},$$

where $r(i)$ represents the i-th entry in the row, and let

$$p = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ p(0) & p(1) & \cdots & p(n-1) \end{pmatrix}.$$

It is easy to see that if p is applied to the columns of $\boldsymbol{R}$, then r is replaced by $r \circ p^{-1}$. Similarly, if p is applied to the symbol set of $\boldsymbol{R}$, then r is replaced by $p \circ r$. Hence, if p is applied to the columns and the symbol set of $\boldsymbol{R}$, then r is replaced by $p \circ r \circ p^{-1}$, known as a *conjugate permutation* of r, which, according to [2, p. 80, Lemma 3.13], necessarily has the same cycle structure as r. In fact, for any two permutations p_1 and p_2 with the same cycle structure there always exists a permutation q such that $q \circ p_1 \circ q^{-1} = p_2$. Since a permutation and its inverse have the same cycle structure, the non-empty rows of $\boldsymbol{R}_{(1,3,2)}$ and $\boldsymbol{R}$ also exhibit the same cycle structure. Hence the cycle structures of the non-empty rows of a RSLR are invariant under a pseudo-isotopism.

Since the diagonal of a SOLS $\boldsymbol{L}$ is a transversal and $\boldsymbol{L}(i,j) \neq \boldsymbol{L}(j,i)$, the cycle structure of the rows of $\boldsymbol{L}_{(3,2,1)}$ must contain exactly one fixed point and no two-cycles. Furthermore, it is easy to see that the first row of a candidate is a cycle structure representative and that the cycle structure of each row is lexicographically larger than that of the first row. Hence, for each cycle structure there is only one universal for the first element and at any point during the tree search described in Section 2, if a row is added to a candidate such that the cycle structure of that row is lexicographically smaller than the cycle structure of the first row, the corresponding partially completed SOLS may be discarded.

Let the first row of an RSLR $\boldsymbol{R}$ be a cycle structure representative and let the cycle structure of each row be lexicographically larger than that of the first row. Suppose we wish to test whether an RSLR $\boldsymbol{R}$ is a candidate. We know at least that if there exists a lexicographically smaller RSLR, it has the same first row as $\boldsymbol{R}$, and we want to consider pseudo-isotopisms of $\boldsymbol{R}$ for which the first row remains intact. Let $r_1, r_2, \ldots, r_\ell$

be the non-empty rows in $\boldsymbol{R}$ with the same cycle structure as the first row. A pseudo-isotopism of $\boldsymbol{R}$ for which the first row remains intact thus either maps the first row to itself or maps one of these rows to the first row[4]. Therefore, these are the only pseudo-isotopisms we have to consider, and if any of these map $\boldsymbol{R}$ to a lexicographically smaller RSLR, then $\boldsymbol{R}$ is not a candidate.

To find the permutations that map a row r_1 to another row r_2 with the same cycle structure, we consider only those permutations that map each cycle of r_1 to a corresponding cycle of r_2. If there are k cycles of equal length in both r_1 and r_2, there are $k!$ possible mappings of these cycles in r_1 to those in r_2. If the cycle $(x_1, x_2, \ldots, x_m)$ is to be mapped to the cycle $(y_1, y_2, \ldots, y_m)$, we may choose to map x_1 to any y_i and the remainder of the mapping will be defined uniquely. If there are a_i cycles of length i for any $i \leq n$ where n is the order of the permutations, then there are $\prod_{i=1}^{n} a_i! i^{a_i}$ mappings from r_1 to r_2.

The results for the enumeration of the isotopy classes via this approach are given in Table 3.3 together with the required computing times.

n	$\Psi(n)$	Time
4	1	0.0
5	1	0.0
6	0	0.0
7	4	1.7
8	4	72.5
9	175	2:19:33:07.4

Table 3.3: The number, $\Psi(n)$, of isotopy classes of SOLS of order $4 \leq n \leq 9$ together with the required computing times.

4 Enumeration of the isomorphism classes

Two SOLS $\boldsymbol{L}$ and $\boldsymbol{L}'$ are *isomorphic* if a single permutation performed on the rows, columns and symbol set of $\boldsymbol{L}$ or $\boldsymbol{L}^T$ results in $\boldsymbol{L}'$. An *isomorphism* is an element of $S_n \times S_2$ acting on a SOLS and may be denoted by $\boldsymbol{p} = (p_r, p_t)$, where $p_r \in S_n$ is the permutation applied to the rows, columns and symbol set, and $p_t \in S_2$ incorporates matrix transposition. For the purpose of enumerating isomorphism classes we follow the approach of McKay *et al.* [15].

A permutation p of order n is of *type* $(a_1, a_2, \ldots, a_n)$ if it has a_i cycles of length i for $1 \leq i \leq n$. As mentioned in Section 3, if two permutations p_1 and p_2 are of the same type $(a_1, a_2, \ldots, a_n)$, then they are conjugates and there exist $\prod_{i=1}^{n} a_i! i^{a_i}$ permutations q such that $q \circ p_1 \circ q^{-1} = p_2$.

An *autotopism* is an isotopism that maps a SOLS $\boldsymbol{L}$ to itself. Denote by $A(\boldsymbol{L})$ the set of all autotopisms admitted by $\boldsymbol{L}$. For any two isotopisms $\boldsymbol{p} = (p_r, p_s, p_t)$ and $\boldsymbol{q} = (q_r, q_s, q_t)$ the notation $\boldsymbol{pq} = (p_r \circ q_r, p_s \circ q_s, p_t \circ q_t)$ is used to denote the action of first applying $\boldsymbol{q}$ and then $\boldsymbol{p}$. Furthermore, $\boldsymbol{p}^{-1} = (p_r^{-1}, p_s^{-1}, p_t^{-1})$. Let $\mathcal{I}(n)$ denote a set of class representatives, one from each SOLS isotopy class of order n, and let $A'(\boldsymbol{L})$ be the set of autotopisms $\boldsymbol{p}$ of $\boldsymbol{L}$ for which p_r and p_s are both of the same type. For an isotopism $\boldsymbol{p}$ in which p_r and p_s are both of type $(a_1, a_2, \ldots, a_n)$, define the function $\psi(\boldsymbol{p}) = \prod_{i=1}^{n} a_i! i^{a_i}$.

All the isomorphism classes in a single isotopy class are orbits of the group $S_n \times S_2$ and we may find the number of orbits by the Cauchy-Frobenius lemma [11] (sometimes (mistakenly) referred to as Burnside's lemma [17]) which states that, if $F(\boldsymbol{p})$ is the number of SOLS (in a single isotopy class) fixed by $\boldsymbol{p} \in S_n \times S_2$, then the number of isomorphism classes of SOLS in that class is

$$\sum_{\boldsymbol{p} \in S_n \times S_2} \frac{F(\boldsymbol{p})}{|S_n \times S_2|}.$$

[4]Since we map a row to the first row by a permutation on the columns and the symbol set, the permutation on the rows is defined (uniquely) in order to ensure that the resulting RSLR is again idempotent.

In other words, the number of isomorphism classes in a single isotopy class of SOLS is the total number of distinct automorphisms over all SOLS in the class divided by the total number of possible isomorphims. Here two automorphisms are distinct if they are either different permutations or if they are automorphisms of two different SOLS.

Theorem 4.1 *If $F(\boldsymbol{p})$ is the number of SOLS in the isotopy class of a SOLS $\boldsymbol{L}$ fixed by $\boldsymbol{p} \in S_n \times S_2$, then*

$$\sum_{\boldsymbol{p}\in S_n\times S_2} \frac{F(\boldsymbol{p})}{|S_n \times S_2|} = \sum_{\boldsymbol{\alpha}\in A'(\boldsymbol{L})} \frac{\psi(\boldsymbol{\alpha})}{|A(\boldsymbol{L})|}.$$

Proof: Let $\boldsymbol{p}$ be an isotopism and let $\boldsymbol{\alpha} \in A(\boldsymbol{L})$. Then the isotopism $\boldsymbol{p\alpha p}^{-1}$ is an autotopism of some SOLS in the isotopy class of $\boldsymbol{L}$, and any autotopism of a SOLS in the isotopy class of $\boldsymbol{L}$ may be written in this form for some isotopism $\boldsymbol{p}$ and some $\boldsymbol{\alpha} \in A(\boldsymbol{L})$.

For $\boldsymbol{p\alpha p}^{-1}$ to be an automorphism it must hold that $p_r \circ \alpha_r \circ p_r^{-1} = p_s \circ \alpha_s \circ p_s^{-1}$, and hence α_r and α_s must be of the same type. The total number of isotopisms $\boldsymbol{p}$ for which $\boldsymbol{p\alpha p}^{-1}$ is an automorphism is $2n!\psi(\boldsymbol{\alpha})$. This number may be found by noting that p_r may be chosen in $n!$ ways, p_s may be chosen in $\prod_{i=1}^{n} a_i! i^{a_i}$ ways, given that α_r and α_s are of type $(a_1, a_2, \ldots, a_n)$, and finally p_t may be chosen in two ways. We thus count $\sum_{\boldsymbol{\alpha}\in A'(\boldsymbol{L})} 2n!\psi(\boldsymbol{\alpha})$ automorphisms over all SOLS in the isotopy class of $\boldsymbol{L}$, although we may have counted each automorphism more than once.

It is therefore necessary to find the number of equivalence classes of pairs $(\boldsymbol{p}, \boldsymbol{\alpha})$, where $\boldsymbol{p}$ is an isotopism for which $\boldsymbol{p\alpha p}^{-1}$ is an automorphism and $\boldsymbol{\alpha} \in A'(\boldsymbol{L})$. Here two pairs $(\boldsymbol{p}, \boldsymbol{\alpha})$ and $(\boldsymbol{q}, \boldsymbol{\beta})$ are equivalent, denoted by $(\boldsymbol{p}, \boldsymbol{\alpha}) \sim (\boldsymbol{q}, \boldsymbol{\beta})$, if and only if $\boldsymbol{p\alpha p}^{-1} = \boldsymbol{q\beta q}^{-1}$ and $\boldsymbol{q}$ and $\boldsymbol{p}$ both map $\boldsymbol{L}$ to the same SOLS. If we let $\boldsymbol{\gamma} \in A(\boldsymbol{L})$, $\boldsymbol{q} = \boldsymbol{p\gamma}$ and $\boldsymbol{\beta} = \boldsymbol{\gamma}^{-1}\boldsymbol{\alpha\gamma}$, then $\boldsymbol{p\alpha p}^{-1} = \boldsymbol{q\beta q}^{-1}$. Furthermore, $\boldsymbol{\beta} \in A'(\boldsymbol{L})$ and $\boldsymbol{q}$ and $\boldsymbol{p}$ both map $\boldsymbol{L}$ to the same SOLS. We may therefore find, for every $\boldsymbol{\gamma} \in A(\boldsymbol{L})$, a pair $(\boldsymbol{q}, \boldsymbol{\beta})$ such that $(\boldsymbol{p}, \boldsymbol{\alpha}) \sim (\boldsymbol{q}, \boldsymbol{\beta})$, and these equivalence classes have cardinality at least $|A(\boldsymbol{L})|$.

Conversely, let $\boldsymbol{p\alpha p}^{-1} = \boldsymbol{q\beta q}^{-1}$ be automorphisms such that $\boldsymbol{q}$ and $\boldsymbol{p}$ both map $\boldsymbol{L}$ to the same SOLS and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A'(\boldsymbol{L})$. Let $\boldsymbol{\gamma} = \boldsymbol{p}^{-1}\boldsymbol{q} \in A(\boldsymbol{L})$. Thus $\boldsymbol{p\alpha p}^{-1} = \boldsymbol{q\beta q}^{-1} = \boldsymbol{p\gamma\beta\gamma}^{-1}\boldsymbol{p}^{-1}$ and $\boldsymbol{\beta} = \boldsymbol{\gamma}^{-1}\boldsymbol{\alpha\gamma}$. We now have the equivalence found above, and therefore the equivalence classes have size exactly $|A(\boldsymbol{L})|$.

The total number of distinct automorphisms over all SOLS in the isotopy class of $\boldsymbol{L}$ is therefore

$$\sum_{\boldsymbol{\alpha}\in A'(\boldsymbol{L})} \frac{2n!\psi(\boldsymbol{\alpha})}{|A(\boldsymbol{L})|}.$$

Since $|S_n \times S_2| = 2n!$ the result follows. □

To find the number of isomorphism classes of SOLS of order n we sum over all the elements of $\mathcal{I}(n)$. The number of isomorphism classes of SOLS of order n is therefore

$$\sum_{\boldsymbol{L}\in\mathcal{I}(n)} \frac{1}{|A(\boldsymbol{L})|} \sum_{\boldsymbol{\alpha}\in A'(\boldsymbol{L})} \psi(\boldsymbol{\alpha}).$$

We use the isotopy class representatives generated by the method of Section 3 as the set $\mathcal{I}(n)$. The computer program `nauty` [13] may be used to determine the autotopy groups of these SOLS. Since `nauty` takes only graphs as input it is necessary to represent a SOLS $\boldsymbol{L}$ by a graph in such a way that the isomorphisms of the graph corresponds to the isotopisms of $\boldsymbol{L}$. Such a graph for each class of Latin squares is described by McKay *et al.* [15] and we use a similar graph representation approach to find the isotopy class of SOLS.

For a SOLS $\boldsymbol{L}$ of order n, let $G(\boldsymbol{L})$ be a vertex-coloured graph such that $V(G) = \{r_i, c_i, s_i, \ell_{ij} \mid i, j \in \mathbb{Z}_n\} \cup \{R, C\}$, where one colour is assigned to $\{r_i, c_i \mid i \in \mathbb{Z}_n\}$, another to $\{s_i \mid i \in \mathbb{Z}_n\}$, a third to $\{R, C\}$ and a fourth colour to $\{\ell_{ij} \mid i, j \in \mathbb{Z}_n\}$. Furthermore, $E(G) = \{r_i\ell_{ij}, c_j\ell_{ij}, s_k\ell_{ij} \mid \boldsymbol{L}(i,j) = k\} \cup \{Rr_i, Cc_i, r_ic_i \mid i \in \mathbb{Z}_n\}$ and the isomorphisms of $G(\boldsymbol{L})$ are colour-preserving. For illustrative purposes a part of $G(\boldsymbol{L}_4)$ is shown in Figure 4.1, where

$$\boldsymbol{L}_4 = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 3 \end{bmatrix}.$$

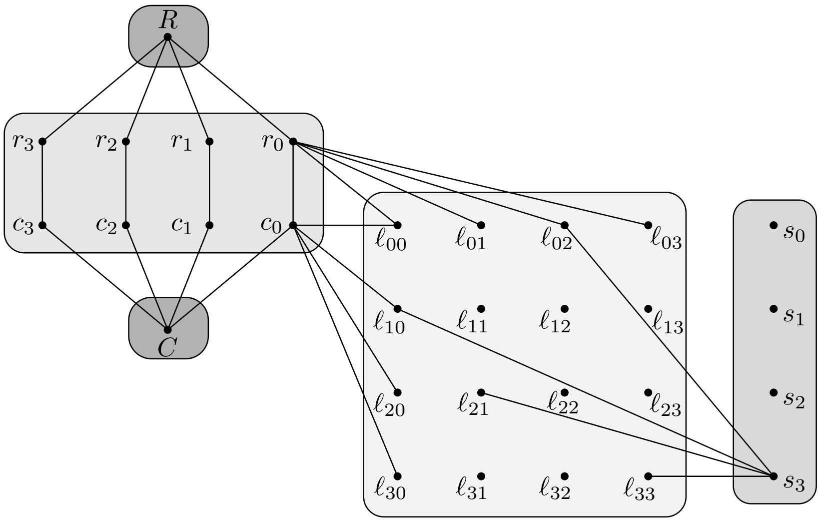

Figure 4.1: A section of the graph $G(\boldsymbol{L}_4)$ with some edges omitted. The isomorphisms of this graph corresponds to the isotopisms of $\boldsymbol{L}_4$.

A SOLS $\boldsymbol{L}$ is *label-isotopic* to a SOLS $\boldsymbol{L}'$ if a single permutation on the symbol set of $\boldsymbol{L}$ results in $\boldsymbol{L}'$. As validation of the correctness of the number of isomorphism classes enumerated via the method described above we employ a second (less efficient) method to enumerate the isomorphism classes for small orders ($4 \leq n \leq 5$). This is achieved by means of the following proposition.

Proposition 4.1 *Any SOLS $\boldsymbol{L}$ is label-isotopic to at least one SOLS from each isomorphism class in the isotopy class of $\boldsymbol{L}$.*

Proof: Let $\mathcal{A}$ be any isomorphism class in the isotopy class of $\boldsymbol{L}$. If $\boldsymbol{p} = (p_1, p_2, t)$ is an isotopism from $\boldsymbol{L}$ to a SOLS in $\mathcal{A}$, then let $\boldsymbol{p}' = (p_1^{-1}, p_1^{-1}, t^{-1})$ be an isomorphism from a SOLS in $\mathcal{A}$ to another SOLS in $\mathcal{A}$. The isotopism $\boldsymbol{p}'\boldsymbol{p}$ is a label-isotopism from $\boldsymbol{L}$ to a SOLS in $\mathcal{A}$. $\Box$

It follows by Proposition 4.1 that we may perform the $n!$ label-isotopisms on a class representative from each isotopy class and group the resulting SOLS into isomorphism classes via isomorphism testing. The test for isomorphism can easily be derived from that of isotopism.

The results obtained via the enumeration of the isomorphism classes are given in Table 4.1 together with the required computing times using the validation method of Proposition 4.1 for orders $4 \leq n \leq 5$. The computation of the autotopism groups via `nauty` was immediate for $4 \leq n \leq 8$ and 0.92 seconds for $n = 9$.

5 Enumeration of unequal and idempotent SOLS

Two SOLS $\boldsymbol{L}$ and $\boldsymbol{L}'$ are *equal* if $\boldsymbol{L}(i,j) = \boldsymbol{L}'(i,j)$ for all i and j. We denote the number of unequal SOLS of order n by $\Lambda(n)$ and the number of idempotent SOLS of order n by $\Phi(n)$.

n	$\Upsilon(n)$	Time
4	5	0.1
5	11	1.1
6	0	—
7	1 986	—
8	52 060	—
9	34 564 884	—

Table 4.1: The number, $\Upsilon(n)$, of isomorphism classes of SOLS of order n together with the required computing times.

Let $\mathcal{T}$ be a maximal set of pairwise isotopic SOLS of order n. Then $\mathcal{T}$ forms an orbit of the group $S_n^2 \times S_2$ and it follows by a well-known property of group actions [1, 6, 11] (sometimes referred to as the orbit-stabiliser theorem) that $|\mathcal{T}| = 2(n!)^2/|A(\boldsymbol{L})|$. Since we have a class representative from each isotopy class in the set $\mathcal{I}(n)$, it follows that $\Lambda(n) = 2(n!)^2 \sum_{\boldsymbol{L}\in\mathcal{I}(n)} 1/|A(\boldsymbol{L})|$. It may be noted that $\Lambda(n) = n!\Phi(n)$ for any $n \in \mathbb{N}$, which is similar to the well-known result that the number of Latin squares of order n is $n!(n-1)!$ times the number of reduced Latin squares[5] of order n. Thus, $\Phi(n) = 2n! \sum_{\boldsymbol{L}\in\mathcal{I}(n)} 1/|A(\boldsymbol{L})|$.

As validation of the enumeration results obtained by the method described above we generated all unequal SOLS of order n and all idempotent SOLS of order n by the method described in Section 2. This was achieved by starting with an empty $n \times n$ square and a partially completed $n \times n$ square with the elements $0, 1, \ldots, n-1$ on the diagonal in order to generate all unequal and idempotent SOLS of order n, respectively.

The enumeration results for $\Lambda(n)$ and $\Phi(n)$ are shown in Table 5.1 together with the required computing times for the depth-first tree search. Where times are omitted, we were not able to complete the enumeration within an acceptable time-frame.

n	$\Lambda(n)$	Time	$\Phi(n)$	Time
4	48	0.1	2	0.0
5	1 440	6.5	12	0.1
6	0	33:56.1	0	0.5
7	19 535 600	—	3 840	4:33.4
8	4 180 377 600	—	103 680	—
9	25 070 769 561 600	—	69 088 320	—

Table 5.1: The number, $\Lambda(n)$, of unequal SOLS of order n and the number, $\Phi(n)$, of idempotent SOLS of order n together with the required computing times.

6 Validation of results

Our final result, the enumeration of different types of SOLS, is shown in Table 6.1 for orders $4 \le n \le 9$. The numbers tabulated here may also be found in [19], and the corresponding sequence reference numbers have been provided.

As a validation of our algorithmic implementations we included the case $n = 6$ in our computations, verifying empirically the known theoretical result that $\Lambda(6) = \Phi(6) = 0$. We also verified computationally that $\Lambda(n) = n!\Phi(n)$ for small orders.

A validation of the correctness of the values tabulated for $\Psi(n)$ is the fact that all idempotent SOLS of order n were enumerated by the autotopism classes of the $\Psi(n)$ non-isotopic SOLS of order n. Not all of

[5] A Latin square is *reduced* if both its first row and first column are in natural order.

n	$\Lambda(n)$	$\Phi(n)$	$\Upsilon(n)$	$\Psi(n)$
4	48	2	5	1
5	1 440	12	11	1
6	0	0	0	0
7	19 353 600	3 840	1 986	4
8	4 180 377 600	103 680	52 060	4
9	25 070 769 561 600	69 088 320	34 564 884	175
	#A160368	#A160367	#A160366	#A160365.

Table 6.1: Enumeration of various classes of SOLS. Here $\Lambda(n)$ denotes the number of unequal SOLS of order n, $\Phi(n)$ the number of idempotent SOLS of order n, $\Upsilon(n)$ the number of isomorphism classes of SOLS of order n and $\Psi(n)$ the number of isotopy classes of SOLS of order n. The identification numbers allocated to the column sequences of the table in Sloane's Online Encyclopedia of Integer Sequences [19] are shown in the last row of the table.

the $\Phi(n)$ idempotent SOLS would have been enumerated if the value of $\Psi(n)$ were too small. Similarly, too large a value of idempotent SOLS would have been enumerated if the value of $\Psi(n)$ were too large. A final validation of the correctness of our numerical results for the isotopy classes is the fact that the two approaches of Section 3, namely completing skeletons and generating class representatives, yielded the same results.

As validation of the isomorphism class enumeration two approaches were again implemented, namely by the autotopism groups of the isotopy class representatives and by Proposition 4.1, both yielding the same results. Finally, validation of the correctness of the values tabulated for $\Lambda(n)$ and $\Phi(n)$ is the fact that both implementations of the method described in Section 2 yielded the same results for $\Lambda(n)$ and $\Phi(n)$ as enumerated by the autotopism classes of the $\Psi(n)$ non-isotopic SOLS of order n.

7 Conclusion

In this paper we enumerated the number of unequal SOLS, the number of idempotent SOLS, the number of isomorphism classes of SOLS and the number of isotopy classes of SOLS of orders $4 \leq n \leq 9$ by a number of different methods for each count, providing a means of validation towards the correctness of these numbers. Furthermore, we were able to build up a repository of all unequal SOLS of orders $4 \leq n \leq 5$, all idempotent SOLS of orders $4 \leq n \leq 7$ and class representatives from each of the isotopy classes of SOLS of orders $4 \leq n \leq 9$, all of which are available online [12].

The first level of the search-tree for SOLS of order 10 has four nodes (utilizing the orderly generation approach) while the second level has 3 825 nodes. In order to gain an understanding of the expected time complexity associated with traversing this search tree, we fathomed 13 (0.34%) of the 3 825 nodes on the second level of the tree. During this process we uncovered 494 non-isotopic SOLS of order 10 (these SOLS are available in [12]). We recorded traversal times ranging between two days and twenty days per node from which we estimate that the time required to traverse the entire tree may be between 20 years and 200 years. Our method is therefore infeasible for order 10, even if a low-level programming language (such as C^{++} or C#) were to be used instead of Mathematica, unless a massively parallel approach is adopted. Further pruning rules are required for early identification of branches of the search tree that will not result in any new SOLS, thus thinning the tree considerably. The exploration of special properties of SOLS of order 10 may also be considered in order to render the search tractable. However, once we have a class representative from each isotopy class of SOLS of a certain order $n \geq 10$, the enumeration of the number of unequal SOLS, idempotent SOLS and isomorphism classes will be neither difficult nor time-consuming.

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Appendix: Non-isotopic SOLS of orders $4 \leq n \leq 9$

All non-isotopic SOLS of orders $4 \leq n \leq 9$ are listed in this appendix. Below each SOLS the number of idempotent SOLS isotopic to it is given in bold face. This value may be multiplied by $n!$ to obtain the cardinality of the corresponding isotopy class.

$n = 4$	$n = 5$	$n = 7$				$n = 8$			
0231	02143	0214365	0214365	0214563	0234561	02145673	02145673	02345671	02345671
3102	31402	3160542	3160524	3105624	6105243	31670245	31702456	31460752	41607352
1320	43210	4526013	4526031	4621035	1420635	43257016	46257301	65207413	53270146
2013	24031	5643201	5643102	6053241	2653014	20736154	57630124	74136205	67431025
2	10324	6351420	6352410	2536410	3516402	65314702	75364210	27654130	76154203
	12	1032654	2401653	1460352	4362150	76402531	13076542	13072546	14726530
		2405136	1035246	5342106	5041326	17523460	20413765	50721364	20513764
		240	**1 680**	**1 680**	**240**	54061327	64521037	46513027	35062417
						80 640	**10 080**	**11 520**	**1 440**

$\boldsymbol{n = 9}$

021436587	021436587	021436587	021436587	021436587	021436587	021436587	021436587	021436587	021436587	021436587	021436587
318507264	318064725	316078245	316578420	317502864	315874260	310257846	318764250	315784062	310274856	316278405	318027456
472061835	472508361	452801763	482167305	482067135	432058716	452803761	472851306	472013856	472068135	452617830	432810765
207358146	280357416	260387154	570321864	576328041	578361024	576382014	560382741	587360421	586317240	580362741	564378012
180642753	107643852	105742836	605842731	658140723	607142835	763148205	706143825	760148235	758640321	768143052	750641823
746185320	764185203	738165402	748605213	764285310	786205143	148675320	187605432	238605714	207185463	204785316	287165340
853724601	835721640	874523610	837014652	835714602	843527601	805714632	854027613	854271603	835702614	837051624	875204631
635810472	653812074	683254071	154283076	140853276	150683472	284560173	635218074	146852370	164853072	645820173	603582174
564273018	546270138	547610328	263750148	203671458	264710358	637021458	243570168	603527148	643521708	173504268	146753208
181 440	**90 720**	**90 720**	**362 880**	**181 440**	**362 880**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**
021436587	021436587	021436587	021436587	021436587	021436587	021436587	021436785	021436785	021436785	021436785	021436785
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582173064	572184036	562178034	562871034	582071463	542873016	562783014	452718036	472068351	432761850	402758361	462587103
170384256	180362745	180324756	180364752	178364205	204358761	206357841	285367140	548371062	564380127	560382147	578314062
765248310	763548120	705841362	705148326	705148326	785642130	783641250	507142863	687240513	607548231	678041253	603748251
208615743	207815463	273685140	237615840	237615840	160785324	140875326	763085412	136725804	178205364	243175806	284605317
834751602	845273601	834257601	874253601	840753612	837124605	857214603	874503621	854103627	843057612	837504612	845271630
453062871	436051872	456012873	456082173	463582071	658210473	638120475	136820574	263584170	280614573	185623074	136820574
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181 440	**90 720**	**181 440**	**181 440**	**181 440**	**181 440**	**181 440**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**
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314058267	318264057	315784206	315782406	315784206	315784026	317508462	314687052	317628504	316870524	318670254	314602857
402671853	432687510	432051867	432057861	432057861	432057861	462810357	482703516	462187350	472058163	532061847	562087314
578324106	574301862	587362041	587361042	587361042	587361240	586372041	540378261	548370162	567384201	174308526	157328046
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237865410	703825146	870625134	846275130	876205134	876205134	134765820	168025437	134865027	283605417	867215403	806175432
850217634	857013624	143578620	173508624	143578620	143578602	870254613	837512640	870512643	835127640	240857631	478513620
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725 760	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**362 880**
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164387502	148367250	185367240	154360827	157364802	160384257	104358267	264380157	247361850	207354861	270384156	286357401
675041823	786540321	763148052	785143260	708542361	756148032	785643021	657048321	608143527	670843512	608541237	765048312
837105264	237085146	278605413	207685143	263085417	407825316	467085132	708125436	834075216	863715204	867125403	470125863
458270631	804753612	854012637	840517632	840157623	845710623	853712640	835271604	183527604	158027643	453078612	807513624
283514076	653812074	640823571	638052471	684213570	683052471	648201573	180654273	450682371	485102376	184652370	134862570
740652318	460271538	436570128	463271508	436721058	274503168	270164358	473562018	765204138	734261058	736210548	653270148
725 760	**362 880**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**
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834605127	184675023	147205836	738605124	740215836	138705426	203875164	478205316	836725140	186705324	178265034	463875120
457283610	457283610	834750621	150283647	153802647	847012653	875014632	804127653	283074651	258074631	235187640	184023657
168520473	630851472	453068172	205864371	468120573	406258371	146250873	180563472	108562374	604258173	684520173	230561874
640712358	763124508	270513468	463571208	204761358	254167038	457623018	237610548	475613208	473512068	463702518	645207318
725 760	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**725 760**	**90 720**	**725 760**	**362 880**	**725 760**	**725 760**
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582071463	542813067	562873401	562087314	502163847	502163847	502614837	532761840	532761840	502768341	532067841	562807431
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875240316	673048512	653047812	685143207	758240361	758042361	738542061	758240361	758042361	738641520	708543126	735648012
436185027	187265403	178205364	473815062	176085234	186275034	176805243	174085236	184275036	283105467	240815367	480175326
208517634	438702651	480751623	204758631	483517602	473518602	483157620	483517602	473518602	450872613	457128603	253781640
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725 760	**120 960**	**725 760**	**725 760**	**362 880**	**362 880**	**362 880**	**181 440**	**181 440**	**362 880**	**725 760**	**725 760**
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514680327	518674230	513678240	513872460	513824067	516827034	518674230	518674230	518627430	517823460	518623047	513867420
652078413	642158307	642153807	642153807	672153804	642178503	642158307	602158347	672158304	682170534	652170834	642570831
786324501	785362014	785312064	765318042	764318520	764381250	785361024	785361024	784361052	764358012	784362510	768314052
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247165830	360715842	370865421	380765124	380765142	380715426	360715842	364715802	360215847	376205841	376805421	386105247
173802654	104587623	204781653	204587631	245087631	278053641	204587613	240587613	205784613	208714653	247018653	204758613
438251076	253801476	168504372	158604273	158602473	153604872	153802476	153802476	143802576	140582376	160584372	130682574
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642570831	642570813	682751403	432786105	452786310	462837501	472681530	472830561	482567103	472168305	452731860	682730451
786314052	768314052	768324510	285371064	208374165	576314820	584376021	604378125	640372851	605384127	608314257	168374025
875241306	875243106	857640132	156247830	680241537	608742135	650847312	768241350	708246315	750246831	835647012	270148563
368105247	386105247	370185246	670835241	873605421	740685312	867135204	187065234	874615032	864735210	780265341	834615207
204758613	204758631	243078651	847520613	145827603	853021647	238710645	853704612	153708624	247801653	174528603	507281634
130862574	130682574	136502874	564108372	536018274	184506273	106524873	540126873	236184570	583610472	263180574	453826170
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314876052	318267054	310286457	316278540	316287450	314876250	314560827	314562807	310562847	318726540	317268450	316572840
652784103	672830415	672518304	602781435	602718534	672180543	682137450	682137450	682137450	652870413	462837105	432168507
268307415	265371840	268371540	258367014	258376041	507362814	740386215	740386215	704386215	867304251	576310824	578314062
130542867	157048362	705642831	730846152	837642105	153248067	875641032	875641032	875641032	170548362	608742531	687240135
847165230	846725103	483705162	847625301	740865312	486715302	168275304	168075324	168075324	786215034	240185367	804735216
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486230571	430682571	134860275	164502873	165024873	268034175	256804173	256804173	256804173	204631875	184503276	265083471
705621348	704516238	546137028	573014268	473501268	735601428	437012568	437210568	437210568	435067128	753624018	740621358
725 760	**725 760**	**725 760**	**362 880**	**362 880**	**362 880**	**725 760**	**725 760**	**362 880**	**362 880**	**90 720**	**725 760**
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314672850	318627450	318627450	318627450	318627450	318627450	318627450	318627450	317628450	315678420	314708256	314572860
482167305	452168307	452168307	452163807	452163807	452863017	452861307	452863107	452863017	432587061	482617305	482167305
547308162	547301862	540371862	547308162	540378162	540371862	547308162	540371862	540371862	608324517	608372514	647308152
765843021	705843126	705843126	705841326	705841326	705148326	705143826	705148326	805147326	160743852	267540831	765843021
803715246	863075241	863705241	860735241	867035241	867035241	863075241	867035241	768035241	874015236	136085427	803715246
258034617	280734615	287034615	283074615	283704615	283714605	280734615	283704615	283714605	587231604	875123640	258034617
136280574	136280574	136280574	136280574	136280574	136280574	136280574	136280574	136280574	243860175	543861072	136280574
670521438	674512038	674512038	674512038	674512038	674502138	674512038	674512038	674502138	756102348	750234168	570621438
362 880	**362 880**	**90 720**	**181 440**	**362 880**	**725 760**	**90 720**	**181 440**	**362 880**	**725 760**	**120 960**	**181 440**
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318527460	318527460	318527460	318527460	318527460	318527460	315064827	315678240	317528460	316508427	317860542	318724065
452168307	452163807	452163807	452863017	452863107	452863107	482710536	402783561	452863017	472180536	452687130	432687510
647301852	647308152	640378152	640371852	647301852	640371852	648327150	658307124	640371852	785321064	704312856	765308124
705843126	705841326	705841326	705148326	705148326	705148326	756842301	763842015	805147326	650847312	673548201	807142356
863075241	860735241	867035241	860735241	867035241	867035241	873105462	847165302	768035241	264735801	840125367	180265437
280734615	283074615	283704615	283714605	283074615	283704615	230578614	284031657	283714605	837214650	238704615	254873601
136280574	136280574	136280574	136280574	136280574	136280574	164283075	130524876	136280574	148063275	586231074	643510872
574612038	574612038	574612038	574602138	574612038	574612038	507631248	576210438	574602138	503672148	165073428	576031248
181 440	**181 440**	**181 440**	**725 760**	**181 440**	**181 440**	**362 880**	**90 720**	**362 880**	**362 880**	**362 880**	**362 880**
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317608245	314680257	314687250	316208547	316287450	316208457	316287450	317280546	315867240	317620845	318574206	315687420
562830417	572038461	502738461	682731450	602738541	682731540	602873541	642873150	682170534	142768503	432761850	432870156
205387164	658307124	658370124	258370164	258370164	258370164	257318064	576301824	867312405	485317026	506328417	607328541
756243801	867142305	867142305	867142305	867142305	867142305	863742105	205148367	208543167	670843251	781642035	861743205
438175026	743865012	743865012	743865012	743865012	743865012	748065312	468725031	734685012	734085162	870415362	748265013
873014652	285713640	285013647	475083621	584013627	574083621	584130627	834517602	543708621	851402637	145087623	580412637
684521370	130524876	130524876	130524876	130524876	130524876	130524876	180634275	150234876	268531470	264803571	156034872
140762538	406271538	476201538	504617238	475601238	405617238	475601238	753062418	476021358	506274318	657230148	274501368
90 720	**362 880**	**181 440**	**362 880**	**90 720**	**181 440**	**45 360**	**120 960**	**120 960**	**90 720**	**181 440**	**45 360**
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315467820	415263807	415263807	415863027	415863027	415863207	415863207	418267305	415863027	415287360	415287360	417580263
472683501	582617430	582617430	582617430	582617430	582617430	582617430	542671830	582417306	562718403	562718403	532604817
781324065	674308251	604378251	604372851	604372851	604372851	604372851	674308251	647302815	684301257	684302517	648372105
157840236	750842316	750842316	730248516	750248316	730248516	750248316	756843012	750648231	750843126	750843126	871243056
806735142	806735142	876035142	876035142	876035142	876035142	876035142	830725146	876035142	837465012	837065241	786015432
234508617	138074625	138704625	158724603	138724605	158704623	138704625	185034627	231784650	208574631	208471635	154837620
568012473	261480573	261480573	261480375	261480573	261480375	261480573	261480573	168520473	146032875	146530872	205468371
640271358	347521068	347521068	347501268	347501268	347521068	347521068	307512468	304271568	371620548	371624058	360721548
90 720	**181 440**	**181 440**	**362 880**	**181 440**	**362 880**	**181 440**	**362 880**	**181 440**	**90 720**	**45 360**	**90 720**
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572618430	572618430	572618430	682517430	672018453	652478130	682517430	682517430	562078413	572184360	182507463	532604817
604372851	604372851	604372851	508362147	587302146	581307246	574308261	504372861	701384526	805327146	576321840	504378162
830247516	850247316	830247516	236740815	760841325	736841025	750842316	750248316	876541032	637548201	651840327	678140253
786035142	786035142	786035142	841075263	234765801	874265301	806735142	876035142	480235167	481635027	864735201	847265301
158724603	138724605	158704623	375408621	805473612	148032657	138074625	138704625	138752604	150872634	208173654	251837640
261480375	261480573	261480375	150824376	158634270	205614873	261480573	261480573	654803271	268051473	340658172	360521874
347501268	347501268	347521068	764231058	341527068	367520418	347621058	347621058	247160358	746203518	735264018	736052418
181 440	**60 480**	**181 440**	**90 720**	**181 440**	**90 720**	**45 360**	**90 720**	**60 480**	**90 720**	**181 440**	**181 440**

023416785	023416785	023416785	023416785	023456781	023456781	023456781
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652178403	832570416	862753401	832671504	632874105	732061845	582617430
786324510	761328540	781324560	764358012	784312056	681374520	601372854
835740162	675143802	657840312	685140237	206541837	207148356	750248316
247065831	386705124	370185246	340785126	870165324	148235067	876035142
371802654	248057631	248071653	208537641	158237640	854702613	138704625
408251376	150684273	136502874	156204873	541608273	560813274	264180573
160537248	407261358	405267138	471062358	467023518	476520138	347521068
10 080	**181 440**	**60 480**	**5 040**	**60 480**	**181 440**	**5 040**